

COMBINATORIAL R -MATRICES FOR KIRILLOV–RESHETIKHIN CRYSTALS OF TYPE $D_n^{(1)}, B_n^{(1)}, A_{2n-1}^{(2)}$

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ABSTRACT. We calculate the image of the combinatorial R -matrix for any classical highest weight element in the tensor product of Kirillov–Reshetikhin crystals $B^{r,k} \otimes B^{1,l}$ of type $D_n^{(1)}, B_n^{(1)}, A_{2n-1}^{(2)}$. The notion of \pm -diagrams is effectively used for the identification of classical highest weight elements in $B^{1,l} \otimes B^{r,k}$.

1. INTRODUCTION

Let $U'_q(\mathfrak{g})$ be the quantum enveloping algebra associated to an affine algebra \mathfrak{g} without derivation. Let V, V' be finite-dimensional $U'_q(\mathfrak{g})$ -modules. Suppose $V \otimes V'$ is irreducible and V, V' have crystal bases B, B' . Then it is known [11, 16] that there exists a unique map R from $B \otimes B'$ to $B' \otimes B$ commuting with any crystal operators e_i and f_i . There also exists an integer-valued function H on $B \otimes B'$, called energy function, satisfying a certain recurrence relation under the action of e_i (see (3.7)).

Combinatorial R -matrices or energy functions play an important role in the affine crystal theory. In the Kyoto path model [9], that realizes the affine highest weight crystal in terms of a semi-infinite tensor product of perfect crystals, the energy function is an essential ingredient for the computation of the affine weight. In the box-ball system [3, 4] or its generalizations [5] in the formulation of crystal bases, the time evolution of the system is defined by using the combinatorial R -matrix. Energy functions are also crucial in the calculation of conserved quantities. In [19] a new connection was revealed between the energy function and the KKR or KSS bijection [13, 14, 15] that gives a one-to-one correspondence between highest weight paths and rigged configurations.

Recently, for all nonexceptional affine types, all KR crystals, crystal bases of Kirillov–Reshetikhin (KR) modules (if they exist), were shown to exist and their combinatorial structures were clarified [17, 20, 18, 1]. Hence, it is natural to consider the problem of obtaining a rule to calculate the combinatorial R -matrix and energy function.

In this paper, for type $D_n^{(1)}, B_n^{(1)}, A_{2n-1}^{(2)}$ we calculate the image of the combinatorial R -matrix for any classical highest weight element in the tensor product of KR crystals $B^{r,k} \otimes B^{1,l}$ (Theorem 4.1). (Note that the first upper index of the second component is 1.) We also obtain the value of the energy function for such elements. Although we get the rule only for highest weight elements, there is an advantage from the computational point of view, since it is always easy to calculate the action of crystal operators e_i, f_i for $i \neq 0$ not only by hand but also by computer. To identify highest weight elements in the image $B^{1,l} \otimes B^{r,k}$ the notion of \pm -diagrams, introduced in [20], is used effectively.

The paper is organized as follows. In Section 2 we briefly review crystals and \pm -diagrams. In Section 3 we recall the KR crystal $B^{r,k}$ for type $D_n^{(1)}$, $B_n^{(1)}$ and $A_{2n-1}^{(2)}$, and the notion of combinatorial R -matrix and energy function. The condition for an element of $B^{r,k} \otimes B^{1,l}$ or $B^{1,l} \otimes B^{r,k}$ to be classically highest is also presented. The main theorem is given in Section 4. In Section 5 we prove a special case of the theorem, and reduction to this case is discussed in Section 6 according to whether r is odd or even.

Acknowledgements. MO was supported by grant JSPS 20540016. The work of RS is supported by the Core Research for Evolutional Science and Technology of Japan Science and Technology Agency.

2. REVIEWS ON CRYSTALS AND \pm -DIAGRAMS

2.1. Crystals. Let \mathfrak{g} stand for a simple Lie algebra or affine Kac–Moody Lie algebra with index set I and $U_q(\mathfrak{g})$ the corresponding quantized enveloping algebra. Axiomatically, a \mathfrak{g} -crystal is a nonempty set B together with maps

$$\begin{aligned} e_i, f_i : B &\rightarrow B \cup \{0\} & \text{for } i \in I, \\ \text{wt} : B &\rightarrow P, \end{aligned}$$

where P is the weight lattice associated to \mathfrak{g} . The maps e_i and f_i are called Kashiwara operators and wt is the weight function. To each crystal one can associate a crystal graph with vertices in B and an arrow colored $i \in I$ from b to b' if $f_i b = b'$ or equivalently $e_i b' = b$. For $b \in B$ and $i \in I$, let

$$\begin{aligned} \varepsilon_i(b) &= \max\{k \in \mathbb{Z}_{\geq 0} \mid e_i^k b \neq 0\}, \\ \varphi_i(b) &= \max\{k \in \mathbb{Z}_{\geq 0} \mid f_i^k b \neq 0\}. \end{aligned}$$

In this paper we only consider crystal bases coming from $U_q(\mathfrak{g})$ -modules. For a complete definition of crystal bases see for example [10, 6].

Let B_1, B_2 be crystals. Then $B_1 \otimes B_2 = \{b_1 \otimes b_2 \mid b_1 \in B_1, b_2 \in B_2\}$ can be endowed with the structure of crystal. The actions of Kashiwara operators and the value of the weight function are given by

$$\begin{aligned} e_i(b_1 \otimes b_2) &= \begin{cases} e_i b_1 \otimes b_2 & \text{if } \varphi_i(b_1) \geq \varepsilon_i(b_2), \\ b_1 \otimes e_i b_2 & \text{if } \varphi_i(b_1) < \varepsilon_i(b_2), \end{cases} \\ f_i(b_1 \otimes b_2) &= \begin{cases} f_i b_1 \otimes b_2 & \text{if } \varphi_i(b_1) > \varepsilon_i(b_2), \\ b_1 \otimes f_i b_2 & \text{if } \varphi_i(b_1) \leq \varepsilon_i(b_2), \end{cases} \\ \text{wt}(b_1 \otimes b_2) &= \text{wt } b_1 + \text{wt } b_2. \end{aligned}$$

The multiple tensor product is defined inductively. In order to compute the action of e_i, f_i on multiple tensor products, it is convenient to use the rule called “signature rule” [12, 16]. Let $b_1 \otimes b_2 \otimes \cdots \otimes b_m$ be an element of the tensor product of crystals $B_1 \otimes B_2 \otimes \cdots \otimes B_m$. One wishes to find the indices j, j' such that

$$\begin{aligned} e_i(b_1 \otimes \cdots \otimes b_m) &= b_1 \otimes \cdots \otimes e_i b_j \otimes \cdots \otimes b_m, \\ f_i(b_1 \otimes \cdots \otimes b_m) &= b_1 \otimes \cdots \otimes f_i b_{j'} \otimes \cdots \otimes b_m. \end{aligned}$$

To do it, we introduce (i -)signature by

$$\underbrace{\varepsilon_i(b_1)}_{-\dots-} \underbrace{\varphi_i(b_1)}_{+\dots+} \underbrace{\varepsilon_i(b_2)}_{-\dots-} \underbrace{\varphi_i(b_2)}_{+\dots+} \dots \underbrace{\varepsilon_i(b_m)}_{-\dots-} \underbrace{\varphi_i(b_m)}_{+\dots+}.$$

We then reduce the signature by deleting the adjacent $+-$ pair successively. Eventually we obtain a reduced signature of the following form.

$$- - \cdots - + + \cdots +$$

Then the action of e_i (resp. f_i) corresponds to changing the rightmost $-$ to $+$ (resp. leftmost $+$ to $-$). If there is no $-$ (resp. $+$) in the signature, then the action of e_i (resp. f_i) should be set to \emptyset . The value of $\varepsilon_i(b)$ (resp. $\varphi_i(b)$) is given by the number of $-$ (resp. $+$) in the reduced signature.

Consider, for instance, an element $b_1 \otimes b_2 \otimes b_3$ of the 3 fold tensor product $B_1 \otimes B_2 \otimes B_3$. Suppose $\varepsilon_i(b_1) = 1, \varphi_i(b_1) = 3, \varepsilon_i(b_2) = 1, \varphi_i(b_2) = 1, \varepsilon_i(b_3) = 2, \varphi_i(b_3) = 1$. Then the signature and reduced one read

$$\begin{array}{ll} \text{sig} & - + + \cdot - + \cdot - - + \\ \text{red sig} & - \cdot \cdot + \end{array}$$

Thus we have

$$\begin{aligned} e_i(b_1 \otimes b_2 \otimes b_3) &= e_i b_1 \otimes b_2 \otimes b_3, \\ f_i(b_1 \otimes b_2 \otimes b_3) &= b_1 \otimes b_2 \otimes f_i b_3. \end{aligned}$$

We denote by $B(\Lambda)$ the highest weight crystal of highest weight Λ , where Λ is a dominant integral weight. Let Λ_i with $i \in I$ be the fundamental weights associated to a simple Lie algebra. In this paper, we consider the types of B_n, C_n and D_n . As usual, a dominant integral weight $\Lambda = \Lambda_{i_1} + \cdots + \Lambda_{i_k}$ is identified with a partition or Young diagram with columns of height i_j for $1 \leq j \leq k$, except when Λ_{i_j} is a spin weight, namely, Λ_n for type B_n and Λ_{n-1} and Λ_n for type D_n . To represent elements of $B(\Lambda)$ we use Kashiwara–Nakashima (KN) tableaux, a generalization of semistandard Young tableaux for type A_n . For KN tableaux refer to [12]. (See also [1] for a summary.) Contrary to the original one, we use the French notation where parts are drawn in increasing order from top to bottom.

To calculate the actions of e_i, f_i on a KN tableau it is convenient to use so-called the Japanese reading word of a tableau. For a KN tableau T move from right to left and on each column move from bottom to top. During this process we read letters, thereby obtaining a word $w(T)$. A letter can be identified with an element of $B(\Lambda_1)$, crystal of the vector representation. Hence $w(T)$ can be viewed as an element of $B(\Lambda_1)^{\otimes N}$ with N being the number of nodes in T or length of $w(T)$. Then the action of e_i or f_i is calculated by using the signature rule. We still need to remember the crystal graph of $B(\Lambda_1)$ for type B_n, C_n, D_n , but it is easy as described in [12].

2.2. \pm -diagrams. Let X_n be B_n, C_n or D_n . For a subset $J \subset I$, we say that $b \in B$ is J -highest if $e_i b = 0$ for all $i \in J$. We set $J = \{2, 3, \dots, n\}$. We describe J -highest elements in terms of a notion of \pm -diagram [20]. A \pm -diagram P of shape Λ/λ is a sequence of partitions $\lambda \subset \mu \subset \Lambda$ such that Λ/μ and μ/λ are horizontal strips. We depict this \pm -diagram by the skew tableau of shape Λ/λ in which the cells of μ/λ are filled with the symbol $+$ and those of Λ/μ are filled with the symbol $-$. Write $\Lambda = \text{outer}(P)$ and $\lambda = \text{inner}(P)$ for the outer and inner shapes of the \pm -diagram P . For type C_n we have a further requirement: the outer shape Λ contains columns of height at most n , but the inner shape λ is not allowed to be of height n (hence there are no empty columns of height n). As we have discussed we identify a Young diagram with a weight.

Proposition 2.1. [20] *Let Λ be an X_n weight that does not contain spin weights. Then there is an isomorphism of X_{n-1} -crystals*

$$B_{X_n}(\Lambda) \cong \bigoplus_{\substack{\pm\text{-diagrams } P \\ \text{outer}(P)=\Lambda}} B_{X_{n-1}}(\text{inner}(P)).$$

That is, the multiplicity of $B_{X_{n-1}}(\lambda)$ in $B_{X_n}(\Lambda)$, is the number of \pm -diagrams of shape Λ/λ .

There is a bijection $\Phi : P \mapsto b$ from \pm -diagrams P of shape Λ/λ to the set of J -highest elements b of X_{n-1} -weight λ . For any columns of height n containing $+$, place a column $12 \cdots n$. Otherwise, place $\bar{1}$ in all positions in P that contain a $-$, and fill the remainder of all columns by strings of the form $23 \cdots k$. We move through the columns of b from top to bottom, left to right. Each $+$ in P (starting with the leftmost moving to the right ignoring $+$ at height n) will alter b as we move through the columns. Suppose the $+$ is at height h in P . If one encounters a $\bar{1}$, replace $\bar{1}$ by $\overline{h+1}$. If one encounters a 2 , replace the string $23 \cdots k$ by $12 \cdots hh + 2 \cdots k$.

Example 2.1. Let us consider the following \pm -diagram.

$$P = \begin{array}{|c|c|c|c|c|c|c|c|c|} \hline + & - & & & & & & & \\ \hline & & & + & - & & & & \\ \hline & & & & + & + & - & & \\ \hline & & & & & & & & + \\ \hline \end{array}$$

To obtain $\Phi(P)$ we first draw the tableau

$$\begin{array}{|c|c|c|c|c|c|c|c|c|} \hline 5 & \bar{1} & & & & & & & \\ \hline 4 & 4 & 4 & 4 & \bar{1} & & & & \\ \hline 3 & 3 & 3 & 3 & 3 & 3 & \bar{1} & & \\ \hline 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\ \hline \end{array}.$$

Reading from left there are $+$'s at height 4, 3, 2, 2, 1. Each $+$ alter the above tableau as follows. The 1st $+$ changes the first column as 1234 (reading from bottom), the 2nd and 3rd change the second column as 124 $\bar{4}$, the 4th changes the third column as 124 and the 5th changes the fourth column as 134. Therefore, $\Phi(P)$ is given by

$$\begin{array}{|c|c|c|c|c|c|c|c|c|} \hline 4 & \bar{4} & & & & & & & \\ \hline 3 & 4 & 4 & 4 & \bar{1} & & & & \\ \hline 2 & 2 & 2 & 3 & 3 & 3 & \bar{1} & & \\ \hline 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 \\ \hline \end{array}.$$

For a word $\mathbf{a} = a_1 a_2 \cdots a_m$ let $e_{\mathbf{a}} = e_{a_m} \cdots e_{a_2} e_{a_1}$. We use this convention also for f . Note that the order in $e_{\mathbf{a}}$ is reversed from \mathbf{a} . Next proposition shows how we get to the highest element from a \pm -diagram by applying e_i 's.

Proposition 2.2. *Let P be a \pm -diagram whose outer shape has depth r . Suppose $r \leq n-1$ for B_n , $r \leq n$ for C_n , $r \leq n-2$ for D_n . Let c_i be the number of columns of the outer shape with height i . Let c_i^- (resp. c_i^+) be the number of $-$ (resp. $+$) at height i . Define a word \mathbf{a} by*

$$\mathbf{a} = 1^{a_1} 2^{a_2} \cdots (n-1)^{a_{n-1}} n^{\gamma_{a_n}} (n-1)^{\gamma'_{a_{n-1}}} (n-2)^{a'_{n-2}} \cdots 1^{a'_1},$$

where

$$a_i = \sum_{j=1}^{i-1} c_j^- + (c_i - c_i^+) + \sum_{j=i+1}^n (c_j + c_j^- - c_j^+), \quad a'_i = \sum_{j=1}^i c_j^-,$$

where $\gamma = 2$ for B_n , $\gamma = 1$ for the other cases, and $\gamma' = 0$ for D_n , $\gamma' = 1$ for the other cases. Then $e_{\mathbf{a}}\Phi(P)$ is the highest weight element with highest weight given by its outer shape. Moreover, at each step when we apply $e_i^{a_i}$ or $e_i^{a'_i}$, including $e_n^{a_n}, e_{n-1}^{a'_{n-1}}$, the action is maximal, namely, if we apply $e_i^{a_i+1}$ or $e_i^{a'_i+1}$, the outcome turns out 0.

Proof. Suppose $r \neq n$ for type C_n . We first prove the claim when there is no $+$ in P . Set $d_i = c_i - c_i^-$. Then the Japanese reading word of the tableau corresponding to P is given by

$$\bar{1}^{c_1^-} 2^{d_1} (\bar{2})^{c_2^-} (23)^{d_2} \dots (23 \dots r \bar{1})^{c_r^-} (23 \dots r r + 1)^{d_r}.$$

The 1-signature is just given by $-\varepsilon_1$, where $\varepsilon_1 = c_1 + \sum_{i=2}^r (c_i + c_i^+)$, and there is no need to reduce. Hence one can apply $e_1^{\varepsilon_1}$. Calculating similarly for $i = 2, 3, \dots, n, n-2, \dots, 1$ one always has a simple i -signature of the form $-\varepsilon_i$, and we arrive at the highest weight element as desired.

Next we consider the general case. We prove by induction on N , the number of $+$. If $N = 0$, the claim is proven. Suppose $N > 0$ and let h be the height of the lowest $+$ in P . Let P' be the same \pm -diagram as P except that there are one less $+$'s at height h . Compare the Japanese reading word of the corresponding tableaux of P and P' . The difference is:

$$\begin{aligned} &\text{either (i) there is a subword } w = 12 \dots h h + 2 \dots \text{ in } P \\ &\quad \text{but } w' = 23 \dots h + 1 h + 2 \dots \text{ in } P', \\ &\text{or (ii) there is a letter } \overline{h+1} \text{ in } P \text{ but } \bar{1} \text{ in } P'. \end{aligned}$$

Apart from this difference in two words, there are subwords of the form $23 \dots$ or letters $\bar{1}$ on the left and subwords of the form $12 \dots h' h' + 2 \dots$ or $\overline{h'+1}$ for some $h' \geq h$ on the right. Let us calculate the 1-signatures of both words. They are $-A+B$ for P and $-A+1+B-1$ for P' . (There are no $+-$ pairs.) After applying e_1^{\max} on both P and P' , the 2-signatures also turn out of the form $-A'+B'$ for P and $-A'+1+B'-1$ for P' . The difference is that there is $12 \dots h h + 2 \dots$ or $\overline{h+1}$ in P but $13 \dots h + 1 h + 2 \dots$ or $\bar{2}$ in P' . Similar situations continue until we apply e_h , and after applying e_h^{\max} , the two results coincide. Hence we should have the desired result.

The proof in the case of $r = n$ for type C_n is almost the same. The only difference is that we first treat the case when there is no $+$ in P except at height n , since there is no empty column of height n . Hence we omit the proof. \square

Example 2.2. For a \pm -diagram given in Example 2.1 set $\mathbf{a} = 1^7 2^5 3^5 4^3 5^3 \dots 4^3 3^2 2$. Then, according to the previous proposition $e_{\mathbf{a}}\Phi(P)$ is a highest weight element.

Later in this paper we will need to apply e_1 to a \pm -diagram P . Since $e_1 P$ is no longer J -highest, we have to use a pair of \pm -diagrams (P, p) to consider $\{3, 4, \dots, n\}$ -highest elements. Namely, P represents a J -highest element and p represents a $\{3, 4, \dots, n\}$ -highest element in the X_{n-1} -component whose highest

weight vector corresponds to P . Under this bijection we identify a $\{3, 4, \dots, n\}$ -highest element b with a pair of \pm -diagram (P, p) . To describe the action of e_1 on (P, p) perform the following algorithm:

- (1) Successively run through all $+$ in p from left to right and, if possible, pair it with the leftmost yet unpaired $+$ in P weakly to the left of it.
- (2) Successively run through all $-$ in p from left to right and, if possible, pair it with the rightmost yet unpaired $-$ in P weakly to the left.
- (3) Successively run through all yet unpaired $+$ in p from left to right and, if possible, pair it with the leftmost yet unpaired $-$ in p .

Proposition 2.3. [20, Lemma 5.1] *If there is an unpaired $+$ in p , e_1 moves the rightmost unpaired $+$ in p to P . Else, if there is an unpaired $-$ in P , e_1 moves the leftmost unpaired $-$ in P to p . Else e_1 annihilates (P, p) .*

3. KR CRYSTAL $B^{r,k}$ AND COMBINATORIAL R -MATRIX

Let \mathfrak{g} be an affine Lie algebra of type $D_n^{(1)}$, $B_n^{(1)}$, or $A_{2n-1}^{(2)}$ with the underlying finite-dimensional simple Lie algebra \mathfrak{g}_0 of type $X_n = D_n, B_n$, or C_n , respectively. We label the vertices of the corresponding Dynkin diagram according to [7], so the index set of \mathfrak{g} (resp. \mathfrak{g}_0) is $I = \{0, 1, \dots, n\}$ (resp. $I_0 := I \setminus \{0\} = \{1, 2, \dots, n\}$). In this section we review KR crystals $B^{r,k}$ of type \mathfrak{g} given in [18, 20] for $k \in \mathbb{Z}_{\geq 1}$ and $1 \leq r \leq n-2$ for $D_n^{(1)}$, $1 \leq r \leq n-1$ for $B_n^{(1)}$ and $1 \leq r \leq n$ for $A_{2n-1}^{(2)}$.

As an X_n -crystal, $B^{r,k}$ is given by

$$(3.1) \quad B^{r,k} \cong \bigoplus_{\Lambda} B(\Lambda),$$

Here $B(\Lambda)$ is the X_n -crystal of highest weight Λ and the sum runs over all dominant weights Λ that can be obtained from $s\Lambda_r$ by the removal of vertical dominoes, where Λ_i are the i -th fundamental weights of X_n .

In order to define the actions of e_0 and f_0 we first consider an automorphism σ on the KR crystal $B^{r,k}$. The Dynkin diagrams of type $D_n^{(1)}$, $B_n^{(1)}$, and $A_{2n-1}^{(2)}$ all have an automorphism interchanging nodes 0 and 1. σ corresponds to this Dynkin diagram automorphism. By construction σ commutes with e_i and f_i for $i \in J := \{2, 3, \dots, n\}$. Hence it suffices to define σ on J -highest elements. Because of the bijection Φ from \pm -diagrams to J -highest elements as described in Section 2.2, it suffices to define the corresponding map \mathfrak{S} on \pm -diagrams. Let P be a \pm -diagram of shape Λ/λ . Let $c_i = c_i(\lambda)$ be the number of columns of height i in λ for all $1 \leq i < r$ with $c_0 = k - \lambda_1$. If $i \equiv r-1 \pmod{2}$, then in P , above each column of λ of height i , there must be a $+$ or a $-$. Interchange the number of such $+$ and $-$ symbols. If $i \equiv r \pmod{2}$, then in P , above each column of λ of height i , either there is no sign or a \mp pair. Suppose there are $p_i \mp$ pairs above the columns of height i . Change this to $(c_i - p_i) \mp$ pairs. The result is $\mathfrak{S}(P)$, which has the same inner shape λ as P but a possibly different outer shape.

Let $b \in B^{r,k}$ and $\mathbf{a} = a_1 a_2 \cdots a_\ell$ be such that $e_{\mathbf{a}} b$ is a J -highest element. Then, $\sigma(b)$ is given by

$$(3.2) \quad \sigma(b) = f_{\text{Rev}(\mathbf{a})} \circ \Phi \circ \mathfrak{S} \circ \Phi^{-1} \circ e_{\mathbf{a}}(b),$$

where $\text{Rev}(\mathbf{a}) = a_\ell \cdots a_2 a_1$. The affine crystal operators e_0 and f_0 are then defined as

$$(3.3) \quad e_0 = \sigma \circ e_1 \circ \sigma, \quad f_0 = \sigma \circ f_1 \circ \sigma.$$

If $r = 1$, the structure of the KR crystal turns out simple. An crystal element of $B^{1,l}$ can be identified with one-row KN tableau of length l with letters from i, \bar{i} ($1 \leq i \leq n$) (and 0 for $B_n^{(1)}$). Denoting the number of letters i, \bar{i} or 0 by x_i, \bar{x}_i or x_0 , we have the so-called coordinate representation of $B^{1,l}$ [8, 16].

$$(3.4) \quad B^{1,l} = \begin{cases} \{(x_1, \dots, x_n, \bar{x}_n, \dots, \bar{x}_1) \mid x_n \text{ or } \bar{x}_n = 0, \sum_{i=1}^n (x_i + \bar{x}_i) = l\} & \text{for } D_n^{(1)}, \\ \{(x_1, \dots, x_n, x_0, \bar{x}_n, \dots, \bar{x}_1) \mid x_0 = 0 \text{ or } 1, \sum_{i=1}^n (x_i + \bar{x}_i) + x_0 = l\} & \text{for } B_n^{(1)}, \\ \{(x_1, \dots, x_n, \bar{x}_n, \dots, \bar{x}_1) \mid \sum_{i=1}^n (x_i + \bar{x}_i) = l\} & \text{for } A_{2n-1}^{(2)}. \end{cases}$$

The action of e_i, f_i for $i = 1, 2, \dots, n$ can be calculated as we explained in the last paragraph of section 2.1. The action of e_0, f_0 is given by

$$(3.5) \quad \begin{aligned} e_0 b &= \begin{cases} (x_1, x_2 - 1, \dots, \bar{x}_2, \bar{x}_1 + 1) & \text{if } x_2 > \bar{x}_2, \\ (x_1 - 1, x_2, \dots, \bar{x}_2 + 1, \bar{x}_1) & \text{if } x_2 \leq \bar{x}_2, \end{cases} \\ f_0 b &= \begin{cases} (x_1, x_2 + 1, \dots, \bar{x}_2, \bar{x}_1 - 1) & \text{if } x_2 \geq \bar{x}_2, \\ (x_1 + 1, x_2, \dots, \bar{x}_2 - 1, \bar{x}_1) & \text{if } x_2 < \bar{x}_2. \end{cases} \end{aligned}$$

We list the values of ε_i, φ_i below.

$$(3.6) \quad \begin{aligned} \varepsilon_0(b) &= x_1 + (x_2 - \bar{x}_2)_+, \quad \varphi_0(b) = \bar{x}_1 + (\bar{x}_2 - x_2)_+, \\ \varepsilon_i(b) &= \bar{x}_i + (x_{i+1} - \bar{x}_{i+1})_+, \quad \varphi_i(b) = x_i + (\bar{x}_{i+1} - x_{i+1})_+ \text{ if } i \neq 0, n, \\ (\varepsilon_n(b), \varphi_n(b)) &= \begin{cases} (\bar{x}_{n-1} + \bar{x}_n, x_{n-1} + x_n) & \text{for } D_n^{(1)}, \\ (2\bar{x}_n + x_0, 2x_n + x_0) & \text{for } B_n^{(1)}, \\ (\bar{x}_n, x_n) & \text{for } A_{2n-1}^{(2)}, \end{cases} \end{aligned}$$

where $(x)_+ = \max(x, 0)$.

Let us now consider a tensor product of KR crystals $B^{r,k} \otimes B^{r',k'}$. It is known [11, 16] that there exists a unique bijection R , called combinatorial R -matrix, commuting with Kashiwara operators e_i, f_i for any $i = 0, 1, \dots, n$. Since R preserves the weight, $u_0 \otimes u'_0$ should be sent to $u'_0 \otimes u_0$ by R , where u_0 (resp. u'_0) is the I_0 -highest elements of $B(k\Lambda_r)$ (resp. $B(k'\Lambda_{r'})$) in $B^{r,k}$ (resp. $B^{r',k'}$). For the other elements the image is uniquely determined, since $B^{r,k} \otimes B^{r',k'}$ is known to be connected [2]. Next we explain the energy function H . Let $b \otimes b' \in B^{r,k} \otimes B^{r',k'}$ correspond to $\tilde{b}' \otimes \tilde{b} \in B^{r',k'} \otimes B^{r,k}$ by R . Suppose $e_i(b \otimes b') \neq 0$. Applying e_i on both sides of $R(b \otimes b') = \tilde{b}' \otimes \tilde{b}$, we are led to consider the following four cases:

$$\begin{aligned} (\text{LL}) \quad R(e_i b \otimes b') &= e_i \tilde{b}' \otimes \tilde{b}, \\ (\text{LR}) \quad R(e_i b \otimes b') &= \tilde{b}' \otimes e_i \tilde{b}, \\ (\text{RL}) \quad R(b \otimes e_i b') &= e_i \tilde{b}' \otimes \tilde{b}, \\ (\text{RR}) \quad R(b \otimes e_i b') &= \tilde{b}' \otimes e_i \tilde{b}. \end{aligned}$$

Then the function H is uniquely determined, up to adding a constant, by

$$(3.7) \quad H(e_i(b \otimes b')) = \begin{cases} H(b \otimes b') + 1 & \text{if } i = 0 \text{ and case (LL) occurs,} \\ H(b \otimes b') - 1 & \text{if } i = 0 \text{ and case (RR) occurs,} \\ H(b \otimes b') & \text{otherwise.} \end{cases}$$

Although it is not obvious that such a function exists, it is shown to exist [11, 16].

Next we investigate conditions for an element of $B^{r,k} \otimes B^{1,l}$ or $B^{1,l} \otimes B^{r,k}$ to be I_0 -highest. Recall the following fundamental fact:

$$(3.8) \quad e_i(b \otimes b') = 0 \text{ if and only if } e_i b = 0 \text{ and } \varepsilon_i(b') \leq \langle h_i, \text{wt } b \rangle.$$

In particular, if $b \otimes b'$ is I_0 -highest, then b has to be I_0 -highest.

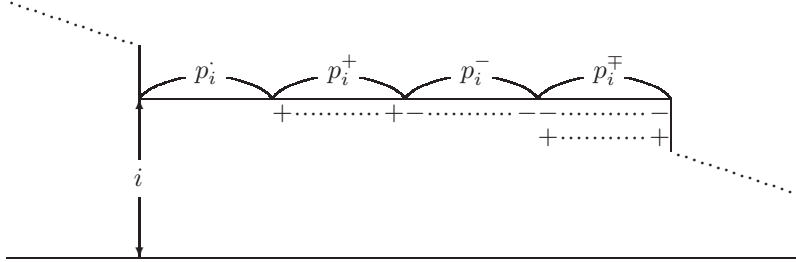
Proposition 3.1. *Let $\mu = \sum_i \mu_i \Lambda_i$ be a dominant integral weight that appears in (3.1) as highest weight. By abuse of notation let μ also stand for the highest KN tableau of weight μ . Let x be an element of $B^{1,l}$ represented by coordinates. Then, an element $b \otimes x$ of $B^{r,k} \otimes B^{1,l}$ is I_0 -highest, if and only if $b = \mu$ for some μ as above and the following conditions for x are satisfied.*

- (i) $x_i = 0$ if $i \geq r + 2$,
- (ii) $\bar{x}_i = 0$ if $i \geq r + 2$ or $i \equiv r + 1 \pmod{2}$,
- (iii) $x_{i+1} + \bar{x}_i \leq \mu_i$ if $i \leq r$ and $i \equiv r \pmod{2}$,
- (iv) $x_i \leq \bar{x}_i$ if $1 < i \leq r$ and $i \equiv r \pmod{2}$.

In the case of $r = n$ where $\mathfrak{g} = A_{2n-1}^{(2)}$, x_{n+1} appearing in (iii) should be understood as 0.

Proof. Apply (3.8) for $i = 1, 2, \dots, n$ and use the formula for ε_i in (3.6). \square

In what follows, for a \pm -diagram P we use the following notation. Let $*$ be one of $\cdot, +, -, \mp$ (\cdot stands for emptiness). We denote by p_i^* the number of columns of the outer shape of P of height i that contain $*$.



Proposition 3.2. *An element $b \otimes b'$ of $B^{1,l} \otimes B^{r,k}$ is I_0 -highest, if and only if $b = 1^l$ and b' is a J -highest element whose corresponding \pm -diagram P satisfies*

$$\sum_{\substack{1 \leq i \leq r \\ i \equiv r \pmod{2}}} (p_i^{\cdot} + p_i^{-}) + \sum_{\substack{1 < i \leq r \\ i \equiv r \pmod{2}}} (p_i^{-} + p_i^{\mp}) \leq l.$$

Proof. Apply (3.8) for $i = 1, 2, \dots, n$ and use Proposition 2.3 to calculate ε_1 of the \pm -diagram P . \square

4. MAIN RESULT

We consider the combinatorial R -matrix

$$R : B^{r,k} \otimes B^{1,l} \longrightarrow B^{1,l} \otimes B^{r,k}.$$

Let $b \otimes x \in B^{r,k} \otimes B^{1,l}$ be I_0 -highest and $R(b \otimes x) = x' \otimes b'$. Then, $x' \otimes b'$ is also I_0 -highest, and from Propositions 3.1 and 3.2 $b = \mu$ for some dominant integral weight μ , $x' = 1^l$ and there exists a \pm -diagram P such that $b' = \Phi(P)$. Thus we have

$$R(\mu \otimes x) = 1^l \otimes \Phi(P).$$

For an element of $B^{1,l}$ we use both the coordinate representation and the Japanese reading word of the corresponding one-row tableau. Let p_i^* ($*$ = $\cdot, +, -, \mp$) be data corresponding to P as in the previous section. Then our main result is:

Theorem 4.1. *With the notations above we have the following formulas.*

$$p_i = \begin{cases} \bar{x}_2 - x_2 - (x_1 - \mu_0)_- & \text{for } i = 0, \\ x_{r+1} & \text{for } i = r, \\ \bar{x}_{i+2} - x_{i+2} & \text{otherwise,} \end{cases} \quad p_i^+ = \mu_i - \bar{x}_i - x_{i+1},$$

$$p_i^- = \begin{cases} \bar{x}_1 & \text{for } i = 1, \\ x_i & \text{otherwise,} \end{cases} \quad p_i^\mp = \begin{cases} \min(\mu_0, x_1) & \text{for } i = 2, \\ x_{i-1} & \text{otherwise.} \end{cases}$$

Here $(x)_- = \min(x, 0)$, $\mu_0 = k - \sum_{i>0} \mu_i$, and $p_0 = k -$ (the first part of the outer shape of P). We also note $i \equiv r \pmod{2}$.

Moreover, the value of the energy function is given by

$$H(\mu \otimes x) = \begin{cases} \lambda_1 - k - l & \text{if } r \text{ is odd,} \\ (\lambda_1 - k)_+ - l & \text{if } r \text{ is even,} \end{cases}$$

if we normalize H in such a way as $H((k^r) \otimes 1^l) = 0$. Here λ_1 is the first part of the partition corresponding to the weight of $\mu \otimes x$.

Solving the formulas for p_i^* with respect to μ and x , we obtain

Corollary 4.2. *The coordinates of the image $\mu \otimes x$ of the inverse of R for an element $1^l \otimes \Phi(P)$ of $B^{1,l} \otimes B^{r,k}$ are given by*

$$\mu_i = \begin{cases} (l - k + \sum_{i \equiv r \pmod{2}} (p_i^+ - p_i^-))_- + p_4^\mp + p_2^+ + p_2^- + p_0 & \text{if } i = 2 \text{ and } r \text{ is even,} \\ p_{i+2}^\mp + p_i^+ + p_i^- + p_{i-2} & \text{if } 0 < i \leq r, i \neq 2 \text{ and } i \equiv r \pmod{2}, \end{cases}$$

$$x_i = \begin{cases} (l - k + \sum_{i \equiv r \pmod{2}} (p_i^+ - p_i^-))_+ + p_2^\mp & \text{if } i = 1 \text{ and } r \text{ is even,} \\ l - k + \sum_{i \equiv r \pmod{2}} (p_i^+ - p_i^-) + p_1^- & \text{if } i = 1 \text{ and } r \text{ is odd,} \\ p_i^- & \text{if } i \neq 1 \text{ and } i \equiv r \pmod{2}, \\ p_{i+1}^\mp & \text{if } i \neq 1 \text{ and } i \not\equiv r \pmod{2}, \end{cases}$$

$$\bar{x}_i = p_i^- + p_{i-2}.$$

Here we should understand $p_{r+2}^\mp = p_r, p_{-1} = 0$.

In what follows in this section we prove Theorem 4.1 by assuming technical propositions in later sections. We give a proof only for type $A_{2n-1}^{(2)}$, since the difference from the other cases is very small as we have seen in Proposition 2.2. Suppose we need to apply e_a with such a word a as

$$a = \cdots (r+1)^\alpha (r+2)^\alpha \cdots n^\alpha (n-1)^\alpha \cdots (r+1)^\alpha \cdots$$

for type $A_{2n-1}^{(2)}$ (see e.g. (6.1)). Then for type $B_n^{(1)}$ we replace it with

$$a = \cdots (r+1)^\alpha (r+2)^\alpha \cdots n^{2\alpha} (n-1)^\alpha \cdots (r+1)^\alpha \cdots,$$

and for type $D_n^{(1)}$

$$a = \cdots (r+1)^\alpha (r+2)^\alpha \cdots n^\alpha (n-2)^\alpha \cdots (r+1)^\alpha \cdots.$$

Consider first the case when r is odd. Suppose μ, x and P are related as in the statement of the theorem. We are to show

$$(4.1) \quad R(\mu \otimes x) = 1^l \otimes \Phi(P).$$

By Proposition 6.10 showing (4.1) is reduced to the case where x is of the form $x = \overline{3}^{\overline{x}_3} 1^{l-\overline{x}_3}$. Applying this proposition again to this case, it is then reduced to the case where $x = 1^l$, since there is no 2 or $\overline{1}$ in x of the previous case. Hence Proposition 5.1 completes the proof of (4.1). (Notice that when $x = 1^l$, $p_i^+ = \mu_i$ for any odd i and the other p_i^* are all zero.) Using these propositions we can calculate H as

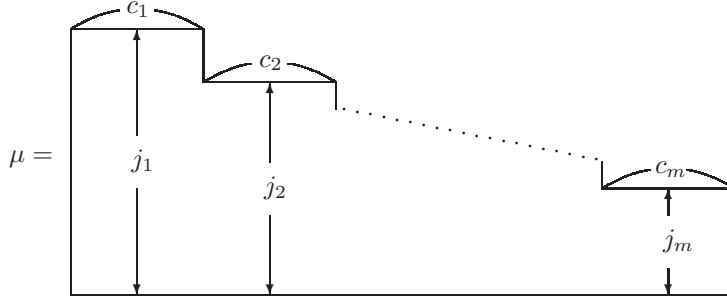
$$\begin{aligned} H(\mu \otimes x) &= H(\overline{\mu} \otimes \overline{3}^{x_2+\overline{x}_1} 1^{l-x_2-\overline{x}_1}) + (x_1 + x_2 - l) \\ &= H(\overline{\mu} \otimes 1^l) + (-x_2 - \overline{x}_1) + (x_1 + x_2 - l) \\ &= x_1 - \overline{x}_1 - l = \lambda_1 - k - l, \end{aligned}$$

since $\lambda_1 = k + x_1 - \overline{x}_1$.

The case when r is even can be proven similarly by using Propositions 6.13 and 5.1.

5. PROOF OF A SPECIAL CASE

5.1. Statement. Let $\mu = \sum_{i=1}^m c_i \Lambda_{j_i}$ be a dominant integral weight whose corresponding Young diagram is depicted as follows.



Here one can assume $j_{i-1} - j_i \in 2\mathbb{Z}_{>0}$ and $c_i > 0$. We also assume that

$$(5.1) \quad \sum_{i=1}^m c_i = k, \quad j_m > 0.$$

Then the claim of this section is the following special version of the main theorem:

Proposition 5.1. *Let μ be as above. Then we have*

$$R(\mu \otimes 1^l) = 1^l \otimes \mu, \quad H(\mu \otimes 1^l) = 0.$$

Remark 5.1. The condition (5.1) for $\mu \in B^{r,k}$ is necessary. For example, in type $D_4^{(1)}$, the image of the combinatorial R -matrix and the value of the energy function for $\begin{smallmatrix} 2 \\ 1 \end{smallmatrix} \otimes \begin{smallmatrix} 1 \end{smallmatrix} \in B^{2,2} \otimes B^{1,1}$ are $\begin{smallmatrix} 1 \end{smallmatrix} \otimes \begin{smallmatrix} 2 & \overline{2} \\ 1 & 2 \end{smallmatrix}$ and -1 .

We divide the proof of this proposition into three parts. Let us define two words that will be used in the proof.

$$\begin{aligned} a_1 &= \underbrace{j_m^{c_m} (j_m + 1)^{c_m} (j_m - 1)^{c_m} j_m^{c_m} (j_m - 2)^{c_m} (j_m - 1)^{c_m} \cdots 1^{c_m} 2^{c_m}}_{a_1}, \\ a_2 &= a_{2,1} a_{2,2} a_{2,3} a_{2,4}, \end{aligned}$$

where underbraces are introduced to show a unit of repetitions, and $\mathbf{a}_{2,1}, \dots, \mathbf{a}_{2,4}$ are defined as follows. Set $\delta = (l - c_m)_+$, then $\mathbf{a}_{2,1} = \mathbf{a}_{2,1,m} \mathbf{a}_{2,1,m-1} \mathbf{a}_{2,1,m-2} \cdots \mathbf{a}_{2,1,2}$ where

$$\begin{aligned} \mathbf{a}_{2,1,m} &= \underbrace{2^{2k+\delta} 1^{2k}} \underbrace{3^{2k+\delta} 2^{2k}} \cdots \underbrace{(j_m + 1)^{2k+\delta} j_m^{2k}}, \\ \mathbf{a}_{2,1,m-1} &= \underbrace{(j_m + 2)^{2k-c_m+\delta} (j_m + 1)^{2k-c_m}} \underbrace{(j_m + 3)^{2k-c_m+\delta} (j_m + 2)^{2k-c_m}} \cdots \\ &\quad \cdots \underbrace{(j_{m-1} - 1)^{2k-c_m+\delta} (j_{m-1} - 2)^{2k-c_m}}, \end{aligned}$$

and for $m - 2 \geq i \geq 2$,

$$\begin{aligned} \mathbf{a}_{2,1,i} &= \underbrace{j_i^{2k-\sum_{j=i}^m c_j+\delta} (j_i - 1)^{2k-\sum_{j=i}^m c_j} (j_i + 1)^{2k-\sum_{j=i}^m c_j+\delta} j_i^{2k-\sum_{j=i}^m c_j}} \cdots \\ &\quad \cdots \underbrace{(j_{i-1} - 1)^{2k-\sum_{j=i}^m c_j+\delta} (j_{i-1} - 2)^{2k-\sum_{j=i}^m c_j}}. \end{aligned}$$

$\mathbf{a}_{2,2}$ is defined by

$$\begin{aligned} \mathbf{a}_{2,2} &= j_1^{k+\delta} (j_1 + 1)^{k+\delta} \cdots n^{k+\delta} (n - 1)^{k+\delta} \cdots j_1^{k+\delta} \\ &\quad \cdot (j_1 - 1)^{2k+\delta} \cdot j_1^k (j_1 + 1)^k \cdots n^k (n - 1)^k \cdots j_1^k \end{aligned}$$

and $\mathbf{a}_{2,3} = \mathbf{a}_{2,3,1} \cdots \mathbf{a}_{2,3,m-2} \mathbf{a}_{2,3,m-1}$ where for $1 \leq i \leq m - 2$,

$$\begin{aligned} \mathbf{a}_{2,3,i} &= \underbrace{(j_i - 2)^{k-\sum_{j=1}^i c_j+\delta} (j_i - 1)^{k-\sum_{j=1}^i c_j} (j_i - 3)^{k-\sum_{j=1}^i c_j+\delta} (j_i - 2)^{k-\sum_{j=1}^i c_j}} \cdots \\ &\quad \cdots \underbrace{(j_{i+1} - 1)^{k-\sum_{j=1}^i c_j+\delta} j_{i+1}^{k-\sum_{j=1}^i c_j}} \end{aligned}$$

and $\mathbf{a}_{2,3,m-1}$ is

$$\begin{aligned} &\underbrace{(j_{m-1} - 2)^{k-\sum_{j=1}^{m-1} c_j+\delta} (j_{m-1} - 1)^{k-\sum_{j=1}^{m-1} c_j} (j_{m-1} - 3)^{k-\sum_{j=1}^{m-1} c_j+\delta} (j_{m-1} - 2)^{k-\sum_{j=1}^{m-1} c_j}} \\ &\quad \cdots \cdots \underbrace{(j_m + 1)^{k-\sum_{j=1}^{m-1} c_j+\delta} (j_m + 2)^{k-\sum_{j=1}^{m-1} c_j}}. \end{aligned}$$

Finally, $\mathbf{a}_{2,4} = j_m^\delta (j_m - 1)^\delta \cdots 2^\delta 1^\delta$.

In the process of proof, we use a dominant integral weight $\bar{\mu}$ given by $\bar{\mu} = \sum_{i=1}^{m-1} c_i \Lambda_{j_i} + c_m \Lambda_{j_m+2}$. We assume that c_m is even. The proof for odd c_m is similar. During the proof, we often identify a KN tableau with its Japanese reading word.

5.2. Proof: Part 1. The goal of this subsection is the following lemma:

Lemma 5.2. $e_{\mathbf{a}_2} e_0^{2k-c_m+\delta} f_{\mathbf{a}_1}(\mu \otimes 1^l) = \bar{\mu} \otimes 1^l$.

This is a direct consequence of the following three sublemmas.

Lemma 5.3.

$$f_{\mathbf{a}_1}(\mu \otimes 1^l) = (34 \cdots (j_m + 2))^{c_m} (12 \cdots j_{m-1})^{c_{m-1}} \cdots (12 \cdots j_2)^{c_2} (12 \cdots j_1)^{c_1} \otimes 1^l.$$

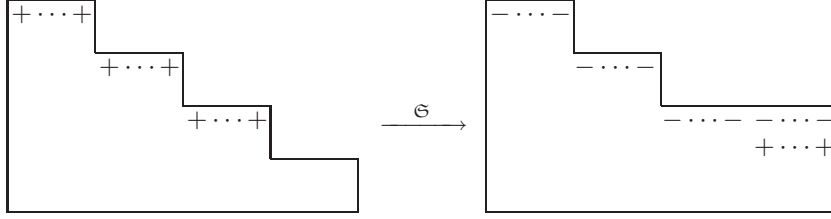
Lemma 5.4.

$$\begin{aligned} e_0^{2k-c_m+\delta} f_{\mathbf{a}_1}(\mu \otimes 1^l) &= (34 \cdots (j_m + 2) \bar{2} \bar{1})^{c_m} (34 \cdots j_{m-1} \bar{2} \bar{1})^{c_{m-1}} \cdots \\ &\quad \cdots (34 \cdots j_2 \bar{2} \bar{1})^{c_2} (34 \cdots j_1 \bar{2} \bar{1})^{c_1} \otimes \bar{2}^\delta 1^{l-\delta}. \end{aligned}$$

Proof. To begin with we apply e_0 on $f_{\mathbf{a}_1}(\mu)$ for maximal times. Define a word \mathbf{a}_3 by $\mathbf{a}_3 = 2^{c_m} 3^{c_m} \cdots (j_m + 1)^{c_m}$. Then the J -highest element of $f_{\mathbf{a}_1}(\mu)$ is

$$e_{\mathbf{a}_3}(f_{\mathbf{a}_1}(\mu)) = (23 \cdots (j_m + 1))^{c_m} (12 \cdots j_{m-1})^{c_{m-1}} \cdots (12 \cdots j_2)^{c_2} (12 \cdots j_1)^{c_1}.$$

By the map Φ^{-1} we get the corresponding \pm -diagram and \mathfrak{S} acts on it as follows:



Note that there are c_m '+'s at height $(j_m + 1)$ of the right \pm -diagram. Assume that c_m satisfies $\sum_{j=1}^{i-1} 2c_j < c_m \leq \sum_{j=1}^i 2c_j$. Then $\Phi \circ \mathfrak{S} \circ \Phi^{-1} \circ e_{\mathbf{a}_3} \circ f_{\mathbf{a}_1}(\mu)$ is

$$\begin{aligned}
 (5.2) \quad & (23 \cdots (j_m + 2)\bar{1})^{c_m} \\
 & (23 \cdots (j_m + 2)(j_m + 3)(j_m + 4) \cdots j_{m-1}\bar{1})^{c_{m-1}} \cdots \\
 & (23 \cdots (j_m + 2)(j_m + 3)(j_m + 4) \cdots j_{i-1}\bar{1})^{c_{i-1}} \\
 & (23 \cdots (j_m + 2)(j_m + 3)(j_m + 4) \cdots j_i\bar{1})^{\sum_{j=1}^i c_j - c_m/2} \\
 & (12 \cdots (j_m + 1)(j_m + 3)(j_m + 4) \cdots j_i \overline{(j_m + 2)})^{c_m/2 - \sum_{j=1}^{i-1} c_j} \\
 & (12 \cdots (j_m + 1)(j_m + 3)(j_m + 4) \cdots j_{i-1} \overline{(j_m + 2)})^{c_{i-1}} \cdots \\
 & (12 \cdots (j_m + 1)(j_m + 3)(j_m + 4) \cdots j_1 \overline{(j_m + 2)})^{c_1}
 \end{aligned}$$

and $f_{\text{Rev}(\mathbf{a}_3)} \circ \Phi \circ \mathfrak{S} \circ \Phi^{-1} \circ e_{\mathbf{a}_3} \circ f_{\mathbf{a}_1}(\mu)$ is

$$\begin{aligned}
 & (234 \cdots (j_m + 2)\bar{1})^{c_m} (234 \cdots j_{m-1}\bar{1})^{c_{m-1}} \cdots (234 \cdots j_{i-1}\bar{1})^{c_i} \\
 & (234 \cdots j_i\bar{1})^{\sum_{j=1}^i c_j - c_m/2} (134 \cdots j_i\bar{2})^{c_m/2 - \sum_{j=1}^{i-1} c_j} \\
 & (134 \cdots j_{i+1}\bar{2})^{c_{i+1}} \cdots (134 \cdots j_2\bar{2})^{c_2} (134 \cdots j_1\bar{2})^{c_1}.
 \end{aligned}$$

From this expression, we get $\varepsilon_0(f_{\mathbf{a}_1}(\mu)) = 2k - c_m$, $\varphi_0(f_{\mathbf{a}_1}(\mu)) = c_m$. Applying $e_1^{\varepsilon_0(f_{\mathbf{a}_1}(\mu))}$ we get

$$\begin{aligned}
 w_1 &:= e_1^{2k-c_m} \circ f_{\text{Rev}(\mathbf{a}_3)} \circ \Phi \circ \mathfrak{S} \circ \Phi^{-1} \circ e_{\mathbf{a}_3} \circ f_{\mathbf{a}_1}(\mu) \\
 &= (134 \cdots (j_m + 2)\bar{2})^{c_m} (134 \cdots j_{m-1}\bar{2})^{c_{m-1}} \cdots (134 \cdots j_2\bar{2})^{c_2} (134 \cdots j_1\bar{2})^{c_1}.
 \end{aligned}$$

To convert the action of $e_1^{2k-c_m}$ into that of $e_0^{2k-c_m}$, we need to define the words $\mathbf{a}_4 = \mathbf{a}_{4,1}\mathbf{a}_{4,2}\mathbf{a}_{4,3}$ as follows. $\mathbf{a}_{4,1} = \mathbf{a}_{4,1,m+1}\mathbf{a}_{4,1,m}\mathbf{a}_{4,1,m-1} \cdots \mathbf{a}_{4,1,2}$ where the subwords $\mathbf{a}_{4,1,i}$ are

$$\begin{aligned}
 \mathbf{a}_{4,1,m+1} &= 2^{2k} 3^{2k} \cdots (j_m + 1)^{2k}, \\
 \mathbf{a}_{4,1,m} &= (j_m + 2)^{2k-c_m} (j_m + 3)^{2k-c_m} \cdots (j_{m-1} - 1)^{2k-c_m}, \\
 \mathbf{a}_{4,1,i} &= j_i^{2k - \sum_{j=i}^m c_j} (j_i + 1)^{2k - \sum_{j=i}^m c_j} \cdots (j_{i-1} - 1)^{2k - \sum_{j=i}^m c_j} \quad (m-1 \geq i \geq 2).
 \end{aligned}$$

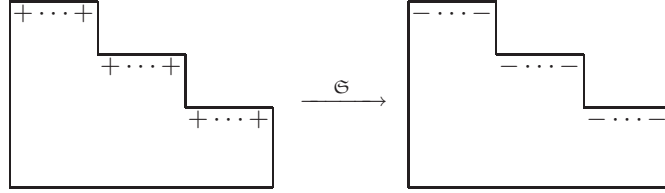
Define $\mathbf{a}_{4,2} = j_1^k(j_1+1)^k \cdots n^k(n-1)^k \cdots (j_1+1)^k j_1^k$ and $\mathbf{a}_{4,3} = \mathbf{a}_{4,3,1} \cdots \mathbf{a}_{4,3,m-1}$ where the subwords $\mathbf{a}_{4,3,i}$ are

$$\begin{aligned} \mathbf{a}_{4,3,i} &= (j_i - 1)^{k - \sum_{j=1}^i c_j} (j_i - 2)^{k - \sum_{j=1}^i c_j} \cdots j_{i+1}^{k - \sum_{j=1}^i c_j}, \quad (2 \leq i \leq m-2), \\ \mathbf{a}_{4,3,m-1} &= (j_{m-1} - 1)^{k - \sum_{j=1}^{m-1} c_j} (j_{m-1} - 2)^{k - \sum_{j=1}^{m-1} c_j} \cdots (j_m + 2)^{k - \sum_{j=1}^{m-1} c_j}. \end{aligned}$$

Then, starting from $e_{\mathbf{a}_{4,1}}(w_1)$, one calculates

$$\begin{aligned} & (123 \cdots (j_m + 1) \overline{j_1})^{c_m} (123 \cdots (j_{m-1} - 1) \overline{j_1})^{c_{m-1}} \cdots (123 \cdots (j_1 - 1) \overline{j_1})^{c_1} \\ & \xrightarrow{e_{\mathbf{a}_{4,2}}} (123 \cdots (j_m + 1) j_1)^{c_m} (123 \cdots (j_{m-1} - 1) j_1)^{c_{m-1}} \cdots (123 \cdots (j_1 - 1) j_1)^{c_1} \\ & \xrightarrow{e_{\mathbf{a}_{4,3}}} (12 \cdots (j_m + 2))^{c_m} (12 \cdots j_{m-1})^{c_{m-1}} \cdots (12 \cdots j_1)^{c_1}. \end{aligned}$$

Here $b \xrightarrow{e_{\mathbf{a}}} b'$ means $e_{\mathbf{a}}b = b'$. With Φ^{-1} , this corresponds to the following \pm -diagram and \mathfrak{S} acts on it as follows:



Thus, starting from $\Phi \circ \mathfrak{S} \circ \Phi^{-1} \circ e_{\mathbf{a}_4}(w_1)$, we calculate

$$\begin{aligned} & (23 \cdots (j_m + 2) \overline{1})^{c_m} (23 \cdots j_{m-1} \overline{1})^{c_{m-1}} \cdots (23 \cdots j_1 \overline{1})^{c_1} \\ & \xrightarrow{f_{\text{Rev}(\mathbf{a}_{4,3})}} (23 \cdots (j_m + 1) j_1 \overline{1})^{c_m} (23 \cdots (j_{m-1} - 1) j_1 \overline{1})^{c_{m-1}} \cdots \\ & \quad \cdots (23 \cdots (j_2 - 1) j_1 \overline{1})^{c_2} (23 \cdots (j_1 - 1) j_1 \overline{1})^{c_1} \\ & \xrightarrow{f_{\text{Rev}(\mathbf{a}_{4,2})}} (23 \cdots (j_m + 1) \overline{j_1} \overline{1})^{c_m} (23 \cdots (j_{m-1} - 1) \overline{j_1} \overline{1})^{c_{m-1}} \cdots \\ & \quad \cdots (23 \cdots (j_2 - 1) \overline{j_1} \overline{1})^{c_2} (23 \cdots (j_1 - 1) \overline{j_1} \overline{1})^{c_1} \\ & \xrightarrow{f_{\text{Rev}(\mathbf{a}_{4,1})}} (34 \cdots (j_m + 2) \overline{2} \overline{1})^{c_m} (34 \cdots j_{m-1} \overline{2} \overline{1})^{c_{m-1}} \cdots (34 \cdots j_2 \overline{2} \overline{1})^{c_2} (34 \cdots j_1 \overline{2} \overline{1})^{c_1}, \end{aligned}$$

where the final formula gives $e_0^{2k-c_m}(f_{\mathbf{a}_1}(\mu))$.

From $\varepsilon_0(f_{\mathbf{a}_1}(\mu))$ and $\varphi_0(f_{\mathbf{a}_1}(\mu))$, we see that the 0-signature of $f_{\mathbf{a}_1}(\mu \otimes 1^l)$ is $-2k-c_m + c_m - l$. Therefore we get

$$\begin{aligned} e_0^{2k-c_m+(l-c_m)+} f_{\mathbf{a}_1}(\mu \otimes 1^l) &= (34 \cdots (j_m + 2) \overline{2} \overline{1})^{c_m} (34 \cdots j_{m-1} \overline{2} \overline{1})^{c_{m-1}} \cdots \\ & \quad \cdots (34 \cdots j_2 \overline{2} \overline{1})^{c_2} (34 \cdots j_1 \overline{2} \overline{1})^{c_1} \otimes \overline{2}^{(l-c_m)+} 1^{l-(l-c_m)+}, \end{aligned}$$

which gives the desired expression. \square

Lemma 5.5. *Starting from $e_{\mathbf{a}_{2,1}} e_0^{2k-c_m+\delta} f_{\mathbf{a}_1}(\mu \otimes 1^l)$, we have*

$$\begin{aligned}
& (123 \cdots j_m \overline{j_1} \overline{(j_1 - 1)})^{c_m} (123 \cdots (j_{m-1} - 2) \overline{j_1} \overline{(j_1 - 1)})^{c_{m-1}} \cdots \\
& (123 \cdots (j_2 - 2) \overline{j_1} \overline{(j_1 - 1)})^{c_2} (123 \cdots (j_1 - 2) \overline{j_1} \overline{(j_1 - 1)})^{c_1} \otimes \overline{j_1}^\delta 1^{l-\delta} \\
& \xrightarrow{e_{\mathbf{a}_{2,2}}} (123 \cdots j_m (j_1 - 1) j_1)^{c_m} (123 \cdots (j_{m-1} - 2) (j_1 - 1) j_1)^{c_{m-1}} \cdots \\
& (123 \cdots (j_2 - 2) (j_1 - 1) j_1)^{c_2} (123 \cdots (j_1 - 2) (j_1 - 1) j_1)^{c_1} \otimes (j_1 - 1)^\delta 1^{l-\delta} \\
& \xrightarrow{e_{\mathbf{a}_{2,3}}} (123 \cdots j_m (j_m + 1) (j_m + 2))^{c_m} (123 \cdots (j_{m-1} - 2) (j_{m-1} - 1) j_{m-1})^{c_{m-1}} \cdots \\
& (123 \cdots (j_2 - 2) (j_2 - 1) j_2)^{c_2} (123 \cdots (j_1 - 2) (j_1 - 1) j_1)^{c_1} \otimes (j_m + 1)^\delta 1^{l-\delta} \\
& \xrightarrow{e_{\mathbf{a}_{2,4}}} (123 \cdots j_m (j_m + 1) (j_m + 2))^{c_m} (123 \cdots (j_{m-1} - 2) (j_{m-1} - 1) j_{m-1})^{c_{m-1}} \cdots \\
& (123 \cdots (j_2 - 2) (j_2 - 1) j_2)^{c_2} (123 \cdots (j_1 - 2) (j_1 - 1) j_1)^{c_1} \otimes 1^l,
\end{aligned}$$

where the final expression is equal to $\bar{\mu} \otimes 1^l$.

Since $e_{\mathbf{a}_{2,4}} e_{\mathbf{a}_{2,3}} e_{\mathbf{a}_{2,2}} e_{\mathbf{a}_{2,1}} = e_{\mathbf{a}_2}$, we have finished the proof of Lemma 5.2.

5.3. Proof: Part 2. The goal of this subsection is the following lemma:

Lemma 5.6. $e_{\mathbf{a}_2} e_0^{2k-c_m+\delta} f_{\mathbf{a}_1}(1^l \otimes \mu) = 1^l \otimes \bar{\mu}$.

To begin with, we have

$$\begin{aligned}
& f_{\mathbf{a}_1}(1^l \otimes (123 \cdots j_m)^{c_m} (123 \cdots j_{m-1})^{c_{m-1}} \cdots (123 \cdots j_1)^{c_1}) \\
& = f_2^{c_m} f_1^{c_m} (1^l \otimes (145 \cdots (j_m + 2))^{c_m} (123 \cdots j_{m-1})^{c_{m-1}} \cdots (123 \cdots j_1)^{c_1}).
\end{aligned}$$

Here, we need to divide the calculation into two cases.

Case 1: If $l < c_m$, we have

$$\begin{aligned}
f_{\mathbf{a}_1}(1^l \otimes \mu) &= 3^l \otimes (345 \cdots (j_m + 2))^{c_m-l} (145 \cdots (j_m + 2))^l \\
&\quad (123 \cdots j_{m-1})^{c_{m-1}} \cdots (123 \cdots j_1)^{c_1}. \\
&=: 3^l \otimes w_2.
\end{aligned}$$

Case 2: If $l \geq c_m$, we have

$$f_{\mathbf{a}_1}(1^l \otimes \mu) = 3^{c_m} 1^{l-c_m} \otimes (145 \cdots (j_m + 2))^{c_m} (123 \cdots j_{m-1})^{c_{m-1}} \cdots (123 \cdots j_1)^{c_1}.$$

5.3.1. Proof for Case 1. To begin with we remark that in this case we have

$$2k - c_m = 2k - c_m + \delta = 2k - c_m + (l - c_m)_+.$$

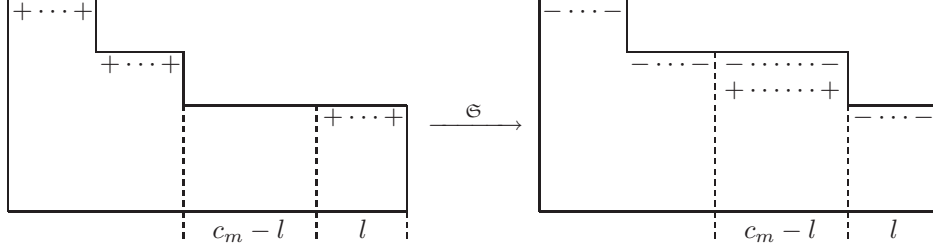
Lemma 5.7. *Assume that l satisfies $\sum_{j=1}^{i-1} 2c_j < l \leq \sum_{j=1}^i 2c_j$. Then*

$$\begin{aligned}
& e_0^{2k-c_m} f_{\mathbf{a}_1}(1^l \otimes \mu) = \\
& 3^l \otimes (345 \cdots (j_m + 2) \overline{2} \overline{1})^{c_m} (345 \cdots j_{m-1} \overline{2} \overline{1})^{c_{m-1}} \cdots (345 \cdots j_{i-1} \overline{2} \overline{1})^{c_{i-1}} \\
& (345 \cdots j_i \overline{2} \overline{1})^{\sum_{j=1}^i c_j - l/2} (145 \cdots j_i \overline{3} \overline{2})^{l/2 - \sum_{j=1}^{i-1} c_j} \\
& (145 \cdots j_{i+1} \overline{3} \overline{2})^{c_{i+1}} \cdots (145 \cdots j_2 \overline{3} \overline{2})^{c_2} (145 \cdots j_1 \overline{3} \overline{2})^{c_1}.
\end{aligned}$$

Proof. Define a word \mathbf{a}_5 by $\mathbf{a}_5 = 2^{c_m-l} \underbrace{3^{c_m} 2^l}_{\substack{\text{---} \\ \text{---}}} \underbrace{4^{c_m} 3^l}_{\substack{\text{---} \\ \text{---}}} \cdots \underbrace{(j_m+1)^{c_m} j_m^l}_{\substack{\text{---} \\ \text{---}}}$. Then we have

$$e_{\mathbf{a}_5}(w_2) = (234 \cdots (j_m+1))^{c_m-l} (123 \cdots j_m)^l (123 \cdots j_{m-1})^{c_{m-1}} \cdots (123 \cdots j_1)^{c_1}.$$

By the map Φ^{-1} , $e_{\mathbf{a}_5}(w_2)$ corresponds to the following \pm -diagram, \mathfrak{S} acts on it as follows:



There are $(c_m - l)$ '+'s at height $(j_m + 1)$ in the right \pm -diagram. Assume that $(c_m - l)$ satisfies $\sum_{j=1}^{i-1} 2c_j < (c_m - l) \leq \sum_{j=1}^i 2c_j$. We also assume that this i satisfies $i < m$ for the sake of simplicity. Then $\Phi \circ \mathfrak{S} \circ \Phi^{-1} \circ e_{\mathbf{a}_5}(w_2)$ is

$$\begin{aligned} & (23 \cdots j_m \bar{1})^l (23 \cdots (j_m + 2) \bar{1})^{c_m-l} \\ & (23 \cdots (j_m + 2)(j_m + 3)(j_m + 4) \cdots j_{m-1} \bar{1})^{c_{m-1}} \cdots \\ & (23 \cdots (j_m + 2)(j_m + 3)(j_m + 4) \cdots j_{i+1} \bar{1})^{c_{i+1}} \\ & (23 \cdots (j_m + 2)(j_m + 3)(j_m + 4) \cdots j_i \bar{1})^{\sum_{j=1}^i c_j - (c_m-l)/2} \\ & (12 \cdots (j_m + 1)(j_m + 3)(j_m + 4) \cdots j_i \overline{(j_m + 2)})^{(c_m-l)/2 - \sum_{j=1}^{i-1} c_j} \\ & (12 \cdots (j_m + 1)(j_m + 3)(j_m + 4) \cdots j_{i-1} \overline{(j_m + 2)})^{c_{i-1}} \cdots \\ & (12 \cdots (j_m + 1)(j_m + 3)(j_m + 4) \cdots j_1 \overline{(j_m + 2)})^{c_1}, \end{aligned}$$

and $f_{\text{Rev}(\mathbf{a}_5)} \circ \Phi \circ \mathfrak{S} \circ \Phi^{-1} \circ e_{\mathbf{a}_5}(w_2)$ is

$$\begin{aligned} & (456 \cdots (j_m + 2) \bar{1})^l (234 \cdots (j_m + 1)(j_m + 2) \bar{1})^{c_m-l} \\ & (234 \cdots (j_m + 1)(j_m + 2)(j_m + 3) \cdots j_{m-1} \bar{1})^{c_{m-1}} \cdots \\ & (234 \cdots (j_m + 1)(j_m + 2)(j_m + 3) \cdots j_{i+1} \bar{1})^{c_{i+1}} \\ & (234 \cdots (j_m + 1)(j_m + 2)(j_m + 3) \cdots j_i \bar{1})^{\sum_{j=1}^i c_j - (c_m-l)/2} \\ & (134 \cdots (j_m + 1)(j_m + 2)(j_m + 3) \cdots j_i \bar{2})^{(c_m-l)/2 - \sum_{j=1}^{i-1} c_j} \\ & (134 \cdots (j_m + 1)(j_m + 2)(j_m + 3) \cdots j_{i-1} \bar{2})^{c_{i-1}} \cdots \\ & (134 \cdots (j_m + 1)(j_m + 2)(j_m + 3) \cdots j_1 \bar{2})^{c_1}. \end{aligned}$$

From this expression, we have $\varepsilon_0(w_2) = 2k - c_m$, $\varphi_0(w_2) = c_m - l$. Applying $e_1^{\varepsilon_0(w_2)}$ we get

$$\begin{aligned} w_3 &:= e_1^{2k-c_m} \circ f_{\text{Rev}(\mathbf{a}_5)} \circ \Phi \circ \mathfrak{S} \circ \Phi^{-1} \circ e_{\mathbf{a}_5}(w_2) \\ &= (456 \cdots (j_m + 2) \bar{2})^l (134 \cdots (j_m + 2) \bar{2})^{c_m-l} (134 \cdots j_{m-1} \bar{2})^{c_{m-1}} \cdots (134 \cdots j_1 \bar{2})^{c_1}. \end{aligned}$$

Note that the length of the string $(456 \cdots (j_m + 2) \bar{2})$ is j_m whereas that of the string $(134 \cdots (j_m + 2) \bar{2})$ is $j_m + 2$.

In order to convert the action of $e_1^{2k-c_m}$ into that of $e_0^{2k-c_m}$, we define the word $\mathbf{a}_6 = \mathbf{a}_{6,1}\mathbf{a}_{6,2}\mathbf{a}_{6,3}\mathbf{a}_{6,4}\mathbf{a}_{6,5}$ as follows. $\mathbf{a}_{6,1} = 2^{2k-l}$, $\mathbf{a}_{6,2} = \mathbf{a}_{6,2,m+1} \cdots \mathbf{a}_{6,2,2}$ where subwords are defined by

$$\begin{aligned} \mathbf{a}_{6,2,m+1} &= 3^{2k} 4^{2k} \cdots (j_m + 1)^{2k}, \\ \mathbf{a}_{6,2,m} &= (j_m + 2)^{2k-c_m} (j_m + 3)^{2k-c_m} \cdots (j_{m-1} - 1)^{2k-c_m}, \\ \mathbf{a}_{6,2,i} &= j_i^{2k-\sum_{j=i}^m c_j} (j_i + 1)^{2k-\sum_{j=i}^m c_j} \cdots (j_{i-1} - 1)^{2k-\sum_{j=i}^m c_j} \quad (m-1 \geq i \geq 2), \end{aligned}$$

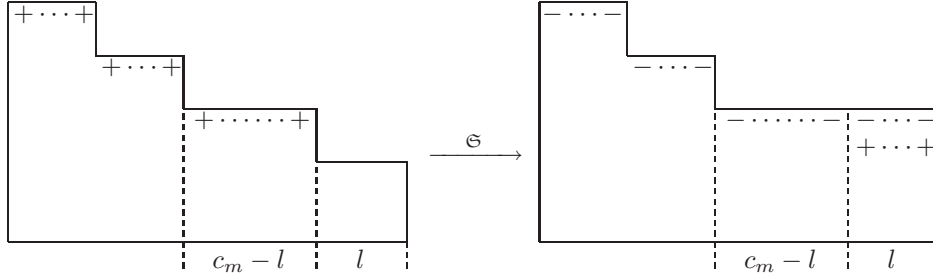
$\mathbf{a}_{6,3} = j_1^k (j_1 + 1)^k \cdots n^k (n-1)^k \cdots (j_1 + 1)^k j_1^k$, $\mathbf{a}_{6,4} = \mathbf{a}_{6,4,1} \cdots \mathbf{a}_{6,4,m-1}$ where subwords are defined by

$$\begin{aligned} \mathbf{a}_{6,4,i} &= (j_i - 1)^{k-\sum_{j=1}^i c_j} (j_i - 2)^{k-\sum_{j=1}^i c_j} \cdots j_{i+1}^{k-\sum_{j=1}^i c_j}, \quad (1 \leq i \leq m-2), \\ \mathbf{a}_{6,4,m-1} &= (j_{m-1} - 1)^{k-\sum_{j=1}^{m-1} c_j} (j_{m-1} - 2)^{k-\sum_{j=1}^{m-1} c_j} \cdots (j_m + 2)^{k-\sum_{j=1}^{m-1} c_j}, \end{aligned}$$

and $\mathbf{a}_{6,5} = (j_m + 1)^l \cdots 3^l 2^l$. Computation of $e_{\mathbf{a}_6}(w_3)$ proceeds as follows:

$$\begin{aligned} e_{\mathbf{a}_{6,1}}(w_3) &= (4567 \cdots (j_m + 2) \bar{3})^l (1245 \cdots (j_m + 2) \bar{3})^{c_m-l} \\ &\quad (1245 \cdots j_{m-1} \bar{3})^{c_{m-1}} \cdots (1245 \cdots j_1 \bar{3})^{c_1} \\ &\xrightarrow{e_{\mathbf{a}_{6,2}}} (345 \cdots (j_m + 1) \bar{j}_1)^l (123 \cdots (j_m + 1) \bar{j}_1)^{c_m-l} \\ &\quad (123 \cdots (j_{m-1} - 1) \bar{j}_1)^{c_{m-1}} \cdots (123 \cdots (j_1 - 1) \bar{j}_1)^{c_1} \\ &\xrightarrow{e_{\mathbf{a}_{6,3}}} (345 \cdots (j_m + 1) j_1)^l (123 \cdots (j_m + 1) j_1)^{c_m-l} \\ &\quad (123 \cdots (j_{m-1} - 1) j_1)^{c_{m-1}} \cdots (123 \cdots (j_1 - 1) j_1)^{c_1} \\ &\xrightarrow{e_{\mathbf{a}_{6,4}}} (345 \cdots (j_m + 1)(j_m + 2))^l (123 \cdots (j_m + 1)(j_m + 2))^{c_m-l} \\ &\quad (123 \cdots (j_{m-1} - 1) j_{m-1})^{c_{m-1}} \cdots (123 \cdots (j_1 - 1) j_1)^{c_1} \\ &\xrightarrow{e_{\mathbf{a}_{6,5}}} (234 \cdots j_m(j_m + 1))^l (123 \cdots (j_m + 1)(j_m + 2))^{c_m-l} \\ &\quad (123 \cdots (j_{m-1} - 1) j_{m-1})^{c_{m-1}} \cdots (123 \cdots (j_1 - 1) j_1)^{c_1}. \end{aligned}$$

By Φ^{-1} , this corresponds to the following \pm -diagram, and \mathfrak{S} acts on it as follows:



Let us assume that $\sum_{j=1}^{i-1} 2c_j < l \leq \sum_{j=1}^i 2c_j$. Then the right \pm -diagram corresponds to the expression (5.2) with $\sum_{j=1}^i c_j - c_m/2$ and $c_m/2 - \sum_{j=1}^{i-1} c_j$ in (5.2) being replaced with $\sum_{j=1}^i c_j - l/2$ and $l/2 - \sum_{j=1}^{i-1} c_j$. Application of $f_{\text{Rev}(\mathbf{a}_{6,5})}$ is similar to that of $f_{\text{Rev}(\mathbf{a}_3)}$ on (5.2) and we obtain $f_{\text{Rev}(\mathbf{a}_{6,5})} \circ \Phi \circ \mathfrak{S} \circ \Phi^{-1} \circ e_{\mathbf{a}_6}(w_3)$

as

(5.3)

$$(234 \dots (j_m + 2)\bar{1})^{c_m} (234 \dots j_{m-1}\bar{1})^{c_{m-1}} \dots (234 \dots j_{i-1}\bar{1})^{c_i} \\ (234 \dots j_i\bar{1})^{\sum_{j=1}^i c_j - l/2} (134 \dots j_i\bar{2})^{l/2 - \sum_{j=1}^{i-1} c_j} \dots (134 \dots j_2\bar{2})^{c_2} (134 \dots j_1\bar{2})^{c_1}.$$

The remaining computation of $f_{\text{Rev}(\mathbf{a}_6)}$ is almost the same as the computation of $f_{\text{Rev}(\mathbf{a}_4)}$ given in the final part of the proof of Lemma 5.4. The only difference in $f_{\mathbf{a}_{6,1}} = f_2^{2k-l}$ is caused by the fact that letters 1 and $\bar{2}$ appear l times in $f_{\text{Rev}(\mathbf{a}_{6,5})} \circ \Phi \circ \mathfrak{S} \circ \Phi^{-1} \circ e_{\mathbf{a}_6}(w_3)$. \square

As for $e_{\mathbf{a}_2}$, the beginning two steps $e_1^{2k} e_2^{2k} e_0^{2k-c_m} f_{\mathbf{a}_1}(1^l \otimes \mu)$ gives

$$1^l \otimes (145 \dots (j_m + 2)\bar{3}\bar{2})^{c_m} (145 \dots j_{m-1}\bar{3}\bar{2})^{c_{m-1}} \dots (145 \dots j_2\bar{3}\bar{2})^{c_2} (145 \dots j_1\bar{3}\bar{2})^{c_1}.$$

By comparing this with $e_1^{2k} e_2^{2k+\delta} e_0^{2k-c_m+\delta} f_{\mathbf{a}_1}(\mu \otimes 1^l)$, we see that the rest of the computation of $e_{\mathbf{a}_2}$ is almost the same as that given in Lemma 5.5. This completes the proof for Case 1.

5.3.2. Proof for Case 2. Note that in this case we have $(l - c_m)_+ = l - c_m$. Action of $e_0^{2k-c_m}$ is obtained by formally setting $l = c_m$ in Case 1. Therefore we have

$$e_0^{2k-c_m+(l-c_m)_+} f_{\mathbf{a}_1}(1^l \otimes \mu) = \bar{2}^{l-c_m} \mathfrak{Z}^{c_m} \otimes (e_0^{2k-c_m}(w_2)|_{l=c_m}).$$

where $e_0^{2k-c_m}(w_2)$ is given in (5.3). When we further apply $e_{\mathbf{a}_2}$ on this formula, we realize that there are extra exponents originating from $\bar{2}^{l-c_m}$ in the first tensor component of the right hand side. These extra contributions coincide with the exponents δ in \mathbf{a}_2 . We have completed the proof of Lemma 5.6.

5.4. Proof: Part 3. Now we can prove Proposition 5.1. We prove the first relation by descending induction on j_m . If $j_m = r$ (this is the maximal possible value), we have $\mu = (k^r)$. In this case we see $((k^r) \otimes 1^l) = 1^l \otimes (k^r)$ by weight consideration. The induction proceeds by using Lemmas 5.2 and 5.6.

As for the energy function, we have to look carefully the action of e_0 in Lemmas 5.2 and 5.6. If e_0 acts on the second component of the tensor product, we write R , and L on the first component. We summarize actions of e_0 to get $\bar{\mu} \otimes 1^l$ and $1^l \otimes \bar{\mu}$ in two lemmas as follows (proceeds from left to right):

$$\mu \otimes 1^l : \underbrace{R \dots \dots \dots R}_{(l-c_m)_+} \underbrace{L \dots \dots L}_{2k-c_m} \\ 1^l \otimes \mu : \underbrace{R \dots \dots R}_{2k-c_m} \underbrace{L \dots \dots \dots L}_{(l-c_m)_+}$$

The diagram is drawn in the case of $(l - c_m)_+ > 2k - c_m$. Including the other inequality case, we see that we have exactly the same number of (LL) and (RR) cases (see (3.7)). Therefore we have $H(\mu \otimes 1^l) = H(\bar{\mu} \otimes 1^l)$. Using the same induction as above we obtain $H(\mu \otimes 1^l) = H((k^r) \otimes 1^l) = 0$. This completes the proof of Proposition 5.1.

6. REDUCTION TO THE SPECIAL CASE

6.1. Odd r case.

6.1.1. *Calculation in $B^{r,k} \otimes B^{1,l}$.* Let $\mu \otimes x \in B^{r,k} \otimes B^{1,l}$ be I_0 -highest. Recall that we defined μ_i by $\mu = \sum_i \mu_i \Lambda_i$. (Readers are warned that it is not the multiplicity of i in the corresponding partition μ but its conjugate μ' .) Note that $\mu_i = 0$ unless $1 \leq i \leq r$ and i is odd. We also know that the coordinates other than $x_1, x_2, \dots, x_{r+1}, \bar{x}_r, \dots, \bar{x}_3, \bar{x}_1$ are all 0 by Proposition 3.1.

Let us define a word $\mathbf{b}_1 = \mathbf{b}_{1,1} \mathbf{b}_{1,2}$ by

$$(6.1) \quad \begin{aligned} \mathbf{b}_{1,1} &= 2^{s_2} 3^{s_3} \dots r^{s_r} (r+1)^\alpha (r+2)^\alpha \dots n^\alpha (n-1)^\alpha \dots (r+1)^\alpha r^\alpha \\ &\quad (r-1)^{\bar{s}_{r-1}} (r-2)^{\bar{s}_{r-2}} \dots 2^{\bar{s}_2}, \\ \mathbf{b}_{1,2} &= 1^{\bar{s}_1} 2^{\bar{s}_2} 3^{\bar{s}_3} \dots (r-1)^{\bar{s}_{r-1}} r^k (r+1)^k \dots n^k (n-1)^k \dots (r+1)^k r^k \\ &\quad (r-1)^{\bar{s}_{r-1}} (r-2)^{\bar{s}_{r-2}} \dots 2^{\bar{s}_2} 1^{\bar{s}_1} \end{aligned}$$

where the exponents are defined as follows. For $i = 1, 2, \dots, (r+1)/2$,

$$s_{2i-1} = 2k - \sum_{j=1:\text{odd}}^{2i-1} \mu_j + x_1 + x_{2i} + \sum_{j=3:\text{odd}}^{2i-1} \bar{x}_j, \quad s_{2i} = s_{2i-1} - x_{2i}.$$

Define $\alpha = k + x_1 + \sum_{j=3:\text{odd}}^r \bar{x}_j$. Then $s_r = \alpha + x_{r+1}$. For $i = (r-1)/2, \dots, 2, 1$,

$$\bar{s}_{2i} = \bar{s}_{2i-1} = \alpha - \sum_{j=2i+1:\text{odd}}^r \mu_j + \sum_{j=2i+1}^{r+1} x_j.$$

Set $\bar{s}_1 = 2k + l - \mu_1$ and define other \bar{s}_i by $\bar{s}_{2i-1} = \bar{s}_{2i} = 2k - \sum_{j=1:\text{odd}}^{2i-1} \mu_j$ for $i = 1, 2, \dots, (r-1)/2$. Note that $\bar{s}_{r-1} = k + \mu_r$. Set $\bar{s}_1 = 0$, $\bar{s}_2 = x_2 + \bar{x}_1$ and define other \bar{s}_i by $\bar{s}_{2i} = \bar{s}_{2i-1} = k - \sum_{j=2i+1:\text{odd}}^r \mu_j$ for $i = (r-1)/2, \dots, 3, 2$.

The goal of this subsection is to prove the following proposition.

Proposition 6.1. *We have*

$$e_{\mathbf{b}_1} e_0^{2k-\mu_1+x_1+x_2} (\mu \otimes x) = \bar{\mu} \otimes \bar{3}^{x_2+\bar{x}_1} 1^{l-x_2-\bar{x}_1}$$

where $\bar{\mu} = \sum_{i=3:\text{odd}}^r \mu_i \Lambda_i + \mu_1 \Lambda_3$.

During the proof, we assume μ_1 even. The proof for odd μ_1 is similar.

Lemma 6.2. *If $\sum_{j=i+1}^r 2\mu_j < \mu_1 \leq \sum_{j=i}^r 2\mu_j$, then*

$$\begin{aligned} e_0^{2k-\mu_1} \mu &= (3\bar{2}\bar{1})^{\mu_1+\mu_3} (345\bar{2}\bar{1})^{\mu_5} \dots (345 \dots (i-2)\bar{2}\bar{1})^{\mu_{i-2}} (345 \dots i\bar{2}\bar{1})^{\sum_{j=i}^r \mu_j - \mu_1/2} \\ &\quad (145 \dots i\bar{3}\bar{2})^{\mu_1/2 - \sum_{j=i+1}^r \mu_j} (145 \dots (i+2)\bar{3}\bar{2})^{\mu_{i+2}} \dots (145 \dots r\bar{3}\bar{2})^{\mu_r}. \end{aligned}$$

Proof. Since $\Phi^{-1}(\mu)$ is the \pm -diagram of outer shape μ such that all the columns have + as symbol, we see $\Phi \circ \mathfrak{S} \circ \Phi^{-1}(\mu) = \bar{1}^{\mu_1} (23\bar{1})^{\mu_3} \dots (23 \dots r\bar{1})^{\mu_r}$. Thus one has $\varepsilon_0(\mu) = 2k - \mu_1$, $\varphi_0(\mu) = 0$. We have $e_1^{2k-\mu_1} \circ \Phi \circ \mathfrak{S} \circ \Phi^{-1}(\mu) = \bar{2}^{\mu_1} (13\bar{2})^{\mu_3} \dots (134 \dots r\bar{2})^{\mu_r}$. To convert the result into that for $e_0^{2k-\mu_1}$ we define a word $\mathbf{b}'_{1,1}$ as follows:

$$\begin{aligned} \mathbf{b}'_{1,1} &= 2^{s'_2} 3^{s'_3} \dots r^{s'_r} (r+1)^{\alpha'} (r+2)^{\alpha'} \dots n^{\alpha'} (n-1)^{\alpha'} \dots (r+1)^{\alpha'} r^{\alpha'} \\ &\quad (r-1)^{s'_{r-1}} (r-2)^{s'_{r-2}} \dots 2^{s'_2} \end{aligned}$$

where $s'_i = s_i|_{x_j=0}$, $\alpha' = \alpha|_{x_j=0}$, $s'_i = \bar{s}_i|_{x_j=0}$. Then we have

$$e_{\mathbf{b}'_{1,1}} \circ e_1^{2k-\mu_1} \circ \Phi \circ \mathfrak{S} \circ \Phi^{-1}(\mu) = 2^{\mu_1} (123)^{\mu_3} (12345)^{\mu_5} \dots (12 \dots r)^{\mu_r}.$$

Applying $\Phi \circ \mathfrak{S} \circ \Phi^{-1}$ further, we obtain

$$(6.2) \quad (23\bar{1})^{\mu_1+\mu_3}(2345\bar{1})^{\mu_5} \dots (23 \dots (i-2)\bar{1})^{\mu_{i-2}}(23 \dots i\bar{1})^{\sum_{j=i}^r \mu_j - \mu_1/2} \\ (1245 \dots i\bar{3})^{\mu_1/2 - \sum_{j=i+1}^r \mu_j} (1245 \dots (i+2)\bar{3})^{\mu_{i+2}} \dots (1245 \dots r\bar{3})^{\mu_r}.$$

Finally, applying $f_{\text{Rev}(\mathbf{b}'_{1,1})}$ we obtain the desired relation. \square

Lemma 6.3. $e_0^{2k-\mu_1+x_1+x_2}(\mu \otimes x)$ is equal to

$$(6.3) \quad e_0^{2k-\mu_1} \mu \otimes \bar{1}^{x_2+\bar{x}_1} \bar{2}^{x_1} \bar{3}^{\bar{x}_3} \dots \overline{(r-2)^{\bar{x}_{r-2}}} \bar{r}^{\bar{x}_r} (r+1)^{x_{r+1}} \dots 4^{x_4} 3^{x_3}$$

Proof. Note that the second component of the RHS is $e_0^{x_1+x_2}x$. Since $\varphi_0(\mu) = 0$, we obtain the expression. \square

Lemma 6.4. $e_{\mathbf{b}_{1,1}} e_0^{2k-\mu_1+x_1+x_2}(\mu \otimes x) = \text{Eq.}(6.2) \otimes \bar{1}^{x_2+\bar{x}_1} 2^{l-x_2-\bar{x}_1}$.

Proof. Let us consider the operation of $e_2^{s_2} = e_2^{2k-\mu_1+x_1}$ in $e_{\mathbf{b}_{1,1}}$. Since $\varphi_2(e_0^{2k-\mu_1}(\mu)) = 0$, e_2 acts on the second component at most $\varepsilon_2(2\text{nd comp})$ times and the rest goes to the first. The 2-signature of the second component of (6.3) is $-x_1+\bar{x}_3-x_3$. From the highest condition for $\mu \otimes x$ we have $\bar{x}_3 \geq x_3$, thus e_2 acts on $\bar{2}^{x_1}$ only. We can continue similarly and obtain the desired result. \square

Finally, we consider the action of $e_{\mathbf{b}_{1,2}}$. The 1-signature of $\text{Eq.}(6.2) \otimes \bar{1}^{x_2+\bar{x}_1} 2^{l-x_2-\bar{x}_1}$ is $-2k-\mu_1(+)-\mu_1/2-l$. By applying $e_1^{2k+l-\mu_1}$, we get

$$(13\bar{2})^{\mu_1+\mu_3}(1345\bar{2})^{\mu_5} \dots (1345 \dots (i-2)\bar{2})^{\mu_{i-2}}(1345 \dots i\bar{2})^{\sum_{j=i}^r \mu_j - \mu_1/2} \\ (1245 \dots i\bar{3})^{\mu_1/2 - \sum_{j=i+1}^r \mu_j} (1245 \dots (i+2)\bar{3})^{\mu_{i+2}} \dots (1245 \dots r\bar{3})^{\mu_r} \otimes \bar{2}^{x_2+\bar{x}_1} 1^{l-x_2-\bar{x}_1}.$$

The 2-signature of the above element is $-2k-\mu_1+\mu_1-x_2+\bar{x}_1$. From the highest condition for $\mu \otimes x$ we have $\mu_1 \geq x_2+\bar{x}_1$, thus e_2 does not act on $\bar{2}^{x_2+\bar{x}_1} 1^{l-x_2-\bar{x}_1}$. Therefore $e_2^{2k-\mu_1}$ acts on the first component and obtain

$$(12\bar{3})^{\mu_1+\mu_3}(1245\bar{3})^{\mu_5} \dots (1245 \dots (i-2)\bar{3})^{\mu_{i-2}}(1245 \dots i\bar{3})^{\sum_{j=i}^r \mu_j - \mu_1/2} \\ (1245 \dots i\bar{3})^{\mu_1/2 - \sum_{j=i+1}^r \mu_j} (1245 \dots (i+2)\bar{3})^{\mu_{i+2}} \dots (1245 \dots r\bar{3})^{\mu_r} \otimes \bar{2}^{x_2+\bar{x}_1} 1^{l-x_2-\bar{x}_1}.$$

We can continue the computation and arrive at Proposition 6.1.

6.1.2. Calculation in $B^{1,l} \otimes B^{r,k}$. In this subsection, let P and P' be the \pm -diagrams. As before, corresponding to P and P' , we use the parametrization p_i^* and $p_i'^*$ ($*$ = $\cdot, +, -, \mp$) respectively. Note that by definition $p_1^\mp = p_1'^\mp = 0$. Define a word $\mathbf{b}'_2 = \mathbf{b}'_{2,1}\mathbf{b}'_{2,2}$ by

$$\mathbf{b}'_{2,1} = 2^{t_2} 3^{t_3} \dots r^{t_r} (r+1)^\beta (r+2)^\beta \dots n^\beta (n-1)^\beta \dots (r+1)^\beta r^\beta \\ (r-1)^{\bar{t}_{r-1}} (r-2)^{\bar{t}_{r-2}} \dots 2^{\bar{t}_2}, \\ \mathbf{b}'_{2,2} = 1^{\bar{t}_1} 2^{\bar{t}_2} 3^{\bar{t}_3} \dots (r-1)^{\bar{t}_{r-1}} r^{\beta'} (r+1)^{\beta'} \dots n^{\beta'} (n-1)^{\beta'} \dots (r+1)^{\beta'} r^{\beta'} \\ (r-1)^{\bar{\bar{t}}_{r-1}} (r-2)^{\bar{\bar{t}}_{r-2}} \dots 2^{\bar{\bar{t}}_2}.$$

Here the exponents for $\mathbf{b}'_{2,1}$ are

$$\begin{aligned} t_2 &= k + \sum_{i=1:\text{odd}}^r (p_i^+ - p_i^-) - p_3^\mp - p_1^+, \\ t_{i+1} - t_i &= \begin{cases} -p_{i+1}^+ & (\text{if } i \text{ is even}) \\ -p_{i+2}^\mp & (\text{if } i \text{ is odd}), \end{cases} \\ \beta - t_r &= -p_r^+, \quad \bar{t}_{r-1} - \beta = -p_r^+ - p_{r-2}^+, \\ \bar{t}_i - \bar{t}_{i+1} &= \begin{cases} -p_{i+1}^+ - p_{i-1}^+ & (\text{if } i \text{ is even}) \\ 0 & (\text{if } i \text{ is odd}). \end{cases} \end{aligned}$$

The exponents for $\mathbf{b}'_{2,2}$ are

$$\begin{aligned} \bar{t}_1 &= 2k - p_3^\mp - p_1^- - p_1^+, \\ \bar{t}_{i+1} - \bar{t}_i &= \begin{cases} -p_{i+3}^\mp - p_{i+1}^- - p_{i+1}^+ - p_{i-1}^+ & (\text{if } i \text{ is even}) \\ 0 & (\text{if } i \text{ is odd}), \end{cases} \\ \beta' - \bar{t}_{r-1} &= \bar{t}_{r-1} - \beta' = -p_{r+2}^\mp - p_r^- - p_r^+ - p_{r-2}^+, \\ \bar{t}_i - \bar{t}_{i+1} &= \begin{cases} -p_{i+3}^\mp - p_{i+1}^- - p_{i+1}^+ - p_{i-1}^+ & (\text{if } i \text{ is even and } i \neq 2) \\ -p_5^\mp - p_3^- - p_3^+ - p_1^+ - p_1^- & (\text{if } i = 2) \\ 0 & (\text{if } i \text{ is odd}). \end{cases} \end{aligned}$$

From \mathbf{b}'_2 , we define another word $\mathbf{b}_2 = \mathbf{b}_{2,1}\mathbf{b}_{2,2}$ as follows:

$$\begin{aligned} \mathbf{b}_{2,1} &= 2^{t_2+l} 3^{t_3+l} \dots r^{t_r+l} (r+1)^{\beta+l} (r+2)^{\beta+l} \dots n^{\beta+l} (n-1)^{\beta+l} \dots (r+1)^{\beta+l} r^{\beta+l} \\ &\quad (r-1)^{\bar{t}_{r-1}+l} (r-2)^{\bar{t}_{r-2}+l} \dots 2^{\bar{t}_2+l}, \\ \mathbf{b}_{2,2} &= 1^{\bar{t}_1+l} 2^{\bar{t}_2} 3^{\bar{t}_3} \dots (r-1)^{\bar{t}_{r-1}} r^{\beta'} (r+1)^{\beta'} \dots n^{\beta'} (n-1)^{\beta'} \dots (r+1)^{\beta'} r^{\beta'} \\ &\quad (r-1)^{\bar{t}_{r-1}} (r-2)^{\bar{t}_{r-2}} \dots 2^{\bar{t}_2}. \end{aligned}$$

Then the goal of this subsection is to prove the following proposition:

Proposition 6.5. *We have*

$$e_{\mathbf{b}_2} e_0^{\varepsilon_0(P)+l} (1^l \otimes P) = 1^l \otimes P'$$

where

$$\varepsilon_0(P) = \sum_j (p_j^- + 2p_j^+ + p_j^\mp) - p_1^+,$$

and P' is related with P as

$$\begin{aligned} p_1' &= p_3^\mp + p_1^-, & p_3'^+ &= p_5^\mp + p_3^- + p_3^+ + p_1^+ + p_1^-, \\ p_r'^+ &= p_r^- + p_r^+ + p_r^\mp + p_{r-2}^+, & p_i'^+ &= p_{i+2}^\mp + p_i^- + p_i^+ + p_{i-2}^+, \end{aligned}$$

where i is an odd integer such that $3 < i < r$ and all other $p_i'^* = 0$.

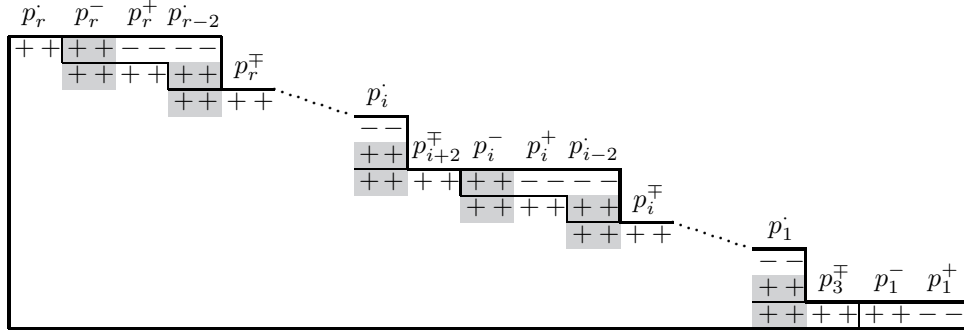
Since $e_0^{\varepsilon_0(P)+l} (1^l \otimes P) = \bar{2}^l \otimes e_0^{\varepsilon_0(P)}(P)$, this proposition is the consequence of the following lemma:

Lemma 6.6. *With the notations of Proposition 6.5, we have $e_{\mathbf{b}_2} e_0^{\varepsilon_0(P)}(P) = P'$.*

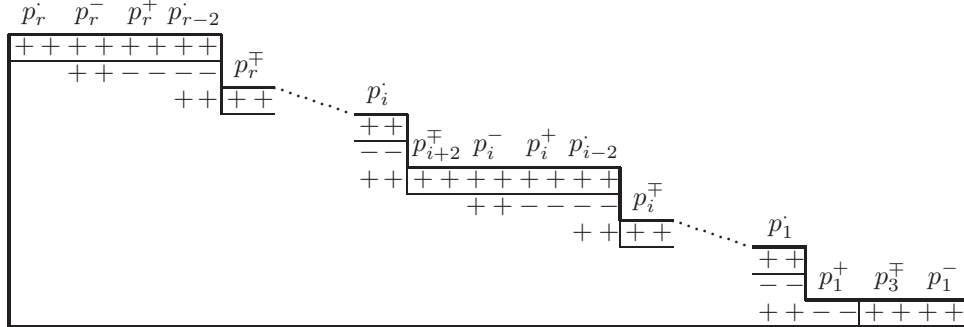
The rest of this subsection is devoted to the proof of this lemma. To begin with, we observe the following:

Lemma 6.7. $\varepsilon_0(P) = \sum_j (p_j + 2p_j^+ + p_j^\mp) - p_1^+$.

Proof. We use Proposition 2.3. Schematically, the pair of \pm -diagrams corresponding to $\mathfrak{S}(P)$ looks as follows:



Here the thick lines represent outer shape of $\mathfrak{S}(P)$ and the thin lines represent the inner \pm -diagram. (Since we are to consider the e_1 action, we need such a pair of \pm -diagrams.) The numbers p_i^* represent the numbers of columns which have the same pattern of $+$ and $-$ indicated below p_i^* . According to Proposition 2.3, we make pairs of two $+$ symbols which we indicate by gray squares in the diagram. Then we see that we can apply e_1 up to $\sum_j (p_j + 2p_j^+ + p_j^\mp) - p_1^+$ times, which gives the value for $\varepsilon_0(P)$. The pair of \pm -diagrams corresponding to $e_1^{\varepsilon_0(P)} \circ \mathfrak{S}(P)$ looks as follows:

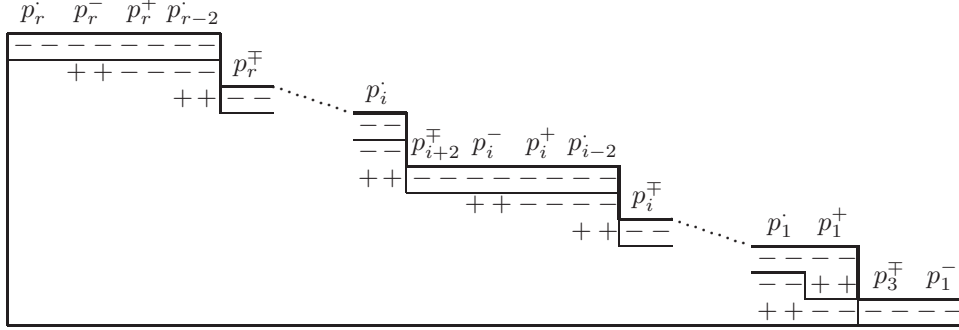


Note that the numbers of columns of height 1 have changed from p_3^\mp, p_1^-, p_1^+ to p_1^+, p_3^\mp, p_1^- . \square

In order to compute $e_0^{\varepsilon_0(P)}(P)$, we usually make $e_1^{\varepsilon_0(P)} \circ \mathfrak{S}(P)$ into $\{2, 3, \dots, n\}$ -highest by applying suitable e_a , apply \mathfrak{S} and then apply $f_{\text{Rev}(a)}$ (see (3.2)). However, since \mathfrak{S} commutes with the action of e_i ($i = 2, 3, \dots, n$), we can apply \mathfrak{S} on the pair of \pm -diagrams directly. Namely, \mathfrak{S} changes the outer \pm -diagram only. The

pair of \pm -diagrams corresponding to $\mathfrak{S} \circ e_1^{\varepsilon_0(P)} \circ \mathfrak{S}(P)$ looks as follows:

(6.4)



Note that the outer shape has also been changed at p_1^+ .

Lemma 6.8. *The inner \pm -diagram of $e_{b'_{2,1}} \circ \mathfrak{S} \circ e_1^{\varepsilon_0(P)} \circ \mathfrak{S}(P)$ is of highest type, i.e., each column contains $+$ as symbol.*

Proof. We use Proposition 2.2. The quantities c_i , c_i^- and c_i^+ there should be used for the corresponding numbers of the inner \pm -diagram of (6.4). Since we are considering the inner \pm -diagram, we have to understand the word \mathbf{a} there as follows:

$$\mathbf{a} = 2^{a_1} 3^{a_2} \dots (n-1)^{a_{n-2}} n^{a_{n-1}} (n-1)^{a'_{n-2}} \dots 2^{a'_1},$$

and the formula for a_i and a'_i are the same in terms of c_i , c_i^- and c_i^+ . Then,

$$\begin{aligned} a_1 &= \sum_{i=1}^n c_i + \sum_{i=1}^n c_i^- - \sum_{i=1}^n c_i^+ - c_1^- \\ &= k + \left\{ \sum_{i=3:\text{odd}}^r (p_i^+ + p_{i-2}) + p_1^+ \right\} - \left\{ \sum_{i=3:\text{odd}}^r (p_i^- + p_{i-2}) + p_3^{\mp} + p_1^- \right\} - p_1^+ \\ &= k + \sum_{i=1:\text{odd}}^r (p_i^+ - p_i^-) - p_3^{\mp} - p_1^+ \end{aligned}$$

and differences $a_{i+1} - a_i$ are

$$\begin{aligned} a_{i+1} - a_i &= c_i^+ + c_i^- - c_i - c_{i+1}^- \\ &= \begin{cases} -p_{i+2}^+ & (\text{if } i \text{ is odd}) \\ -p_{i+3}^{\mp} & (\text{if } i \text{ is even and } i \neq r-1) \\ -p_r^+ & (i = r-1) \end{cases} \end{aligned}$$

and differences $a'_i - a'_{i+1}$ are

$$a'_i - a'_{i+1} = -c_{i+1}^- = \begin{cases} -p_{i+2}^+ - p_i^- & (\text{if } i \text{ is odd}) \\ 0 & (\text{if } i \text{ is even}). \end{cases}$$

We see that the word \mathbf{a} computed here coincides with $\mathbf{b}'_{2,1}$. □

Lemma 6.9. $e_1^{p_3^{\mp} + p_1^-} \circ e_{b'_{2,2}} \circ e_{b'_{2,1}} \circ \mathfrak{S} \circ e_1^{\varepsilon_0(P)} \circ \mathfrak{S}(P)$ is I_0 -highest.

Proof. Again, we use Proposition 2.2. In this case, the quantities c_i , c_i^- and c_i^+ there mean those for the outer \pm -diagram of (6.4). Let us compute the word

$$\mathbf{a} = 1^{a_1} 2^{a_2} \dots (n-1)^{a_{n-1}} n^{a_n} (n-1)^{a'_{n-1}} \dots 1^{a'_1}$$

there in the case of our (6.4). To begin with, a_1 is

$$\begin{aligned} a_1 &= \sum_{i=1}^n c_i + \sum_{i=1}^n c_i^- - \sum_{i=1}^n c_i^+ - c_1^- = k + k - p_1^+ - (p_3^\mp + p_1^-) \\ &= 2k - p_3^\mp - p_1^- - p_1^+, \end{aligned}$$

and differences $a_{i+1} - a_i$ are

$$\begin{aligned} a_{i+1} - a_i &= c_i^+ + c_i^- - c_i - c_{i+1}^- \\ &= \begin{cases} 0 & (\text{if } i \text{ is odd or } i \geq r) \\ -p_{i+3}^\mp - p_{i+1}^- - p_{i+1}^+ - p_{i-1}^- & (\text{if } i \text{ is even and } i \leq r-1) \end{cases} \end{aligned}$$

and differences $a'_i - a'_{i+1}$ are

$$a'_i - a'_{i+1} = -c_{i+1}^- = \begin{cases} 0 & (\text{if } i \text{ is odd or } i \geq r) \\ -p_{i+3}^\mp - p_{i+1}^- - p_{i+1}^+ - p_{i-1}^- & (\text{if } i \text{ is even and } i \neq 2, i \leq r-1) \\ -p_5^\mp - p_3^- - p_3^+ - p_1^- - p_1^+ & (\text{if } i = 2). \end{cases}$$

We see that the word \mathbf{a} computed here coincides with $\mathbf{b}'_{2,2}$ except for $a'_1 = p_3^\mp + p_1^-$ which does not appear in $\mathbf{b}'_{2,2}$. \square

Let $\tilde{\mu}$ be the I_0 -highest weight element whose outer shape coincides with (6.4). Then the above lemma shows that $e_{\mathbf{b}_2} e_0^{\varepsilon_0(P)}(P) = f_1^{p_3^\mp + p_1^-}(\tilde{\mu})$. Since there are exactly $(p_3^\mp + p_1^-)$ columns of height 1 in $\tilde{\mu}$, we see that the content of columns of height 1 in the tableau $f_1^{p_3^\mp + p_1^-}(\tilde{\mu})$ are all 2 and that the other columns are the same as $\tilde{\mu}$. From the shape of (6.4) we see that $f_1^{p_3^\mp + p_1^-}(\tilde{\mu})$ coincides with P' given in Proposition 6.5. To summarize, we have $e_{\mathbf{b}_2} e_0^{\varepsilon_0(P)}(P) = P'$, hence we complete the proof of Proposition 6.5.

We remark that the 1-signature of $1^l \otimes P'$ is $+^l - (p_3^\mp + p_1^-) +^C$ for some C and from the highest weight condition of $1^l \otimes P$ we have $l \geq p_3^\mp + p_1^-$. Thus we cannot apply e_1 on $1^l \otimes P'$.

6.1.3. Reduction to the special case.

Proposition 6.10. *Suppose μ, x and P are related as in Theorem 4.1. Then, with the notions in Propositions 6.1 and 6.5, we have*

- (i) $2k - \mu_1 + x_1 + x_2 = \varepsilon_0(P) + l$,
- (ii) $\mathbf{b}_1 = \mathbf{b}_2$, and
- (iii) In view of Proposition 6.1, set $H = H(\mu \otimes x)$, $H' = H(\bar{\mu} \otimes \bar{3}^{x_2 + \bar{x}_1} 1^{l - x_2 - \bar{x}_1})$. Then we have $H = H' + (x_1 + x_2 - l)$.

Proof. (i) We have

$$\begin{aligned}\varepsilon_0(P) &= \sum_j (p_j^- + 2p_j^+ + p_j^\mp) - p_1^+ = 2k - \sum_j (p_j^- + p_j^\mp + 2p_j^+) - p_1^+ \\ &= 2k - \left(\sum_{j=1:\text{odd}}^{r-2} (\bar{x}_{j+2} - x_{j+2}) + x_{r+1} \right) - \sum_{j=3:\text{odd}}^r x_{j-1} - \left(\sum_{j=3:\text{odd}}^r 2x_j + 2\bar{x}_1 \right) \\ &\quad - (\mu_1 - \bar{x}_1 - x_2) = 2k - l + x_1 - \mu_1 + x_2,\end{aligned}$$

where we have used $\sum_{j=1}^{r+1} x_j + \sum_{j=1:\text{odd}}^r \bar{x}_j = l$ in the final line. Thus $\varepsilon_0(P) + l = 2k - \mu_1 + x_1 + x_2$.

(ii) To begin with let us show $\mathbf{b}_{1,1} = \mathbf{b}_{2,1}$. We compute

$$\begin{aligned}t_2 &= k + \sum_{j=1:\text{odd}}^r (\mu_j - \bar{x}_j - x_{j+1}) - \left(\bar{x}_1 + \sum_{j=3:\text{odd}}^r x_j \right) - x_2 - (\mu_1 - \bar{x}_1 - x_2) \\ &= 2k - l - \mu_1 + x_1,\end{aligned}$$

thus $t_2 + l = s_2$, which shows the coincidence of the first letters of $\mathbf{b}_{1,1}$ and $\mathbf{b}_{2,1}$. As for the other s_i and t_i , note that

$$t_{i+1} - t_i = \begin{cases} -p_{i+1}^+ = -(\mu_{i+1} - \bar{x}_{i+1} - x_{i+2}) & (\text{if } i \text{ is even}) \\ -p_{i+2}^\mp = -x_{i+1} & (\text{if } i \text{ is odd}). \end{cases}$$

When i is odd, we see $t_{i+1} - t_i = s_{i+1} - s_i$. When i is even, we have

$$\begin{aligned}s_{i+1} - s_i &= \left(2k - \sum_{j=1:\text{odd}}^{i+1} \mu_j + x_1 + x_{i+2} + \sum_{j=3:\text{odd}}^{i+1} \bar{x}_j \right) \\ &\quad - \left(2k - \sum_{j=1:\text{odd}}^{i-1} \mu_j + x_1 + \sum_{j=3:\text{odd}}^{i-1} \bar{x}_j \right)\end{aligned}$$

and thus we have $t_{i+1} - t_i = s_{i+1} - s_i$, i.e., $t_i = s_i$ for all i . We have $\beta - t_r = -p_r^- = -x_{r+1} = \alpha - s_r$, i.e., $\alpha = \beta + l$. Similarly, we have $\bar{t}_{r-1} - \beta = -p_r^+ - p_{r-2}^- = -(\mu_r - \bar{x}_r - x_{r+1}) - (\bar{x}_r - x_r) = -\mu_r + x_r + x_{r+1} = \bar{s}_{r-1} - \alpha$. As for other \bar{t}_i and \bar{s}_i , we have

$$\bar{t}_{i+1} - \bar{t}_i = \begin{cases} -p_{i+1}^+ - p_{i-1}^- = -(\mu_{i+1} - \bar{x}_{i+1} - x_{i+2}) - (\bar{x}_{i+1} - x_{i+1}) & (\text{if } i \text{ is even}) \\ 0 & (\text{if } i \text{ is odd}). \end{cases}$$

Thus we have $\bar{t}_i + l = \bar{s}_i$ for all i and obtain $\mathbf{b}_{1,1} = \mathbf{b}_{2,1}$.

Similarly we can show $\mathbf{b}_{1,2} = \mathbf{b}_{2,2}$. We compute

$$\bar{\bar{t}}_1 = 2k - x_2 - \bar{x}_1 - (\mu_1 - \bar{x}_1 - x_2) = 2k - \mu_1,$$

thus $\bar{\bar{t}}_1 + l = \bar{\bar{s}}_1$, i.e., the coincidence of the first letters of $\mathbf{b}_{1,1}$ and $\mathbf{b}_{2,1}$. Next, we have $\bar{\bar{t}}_2 - (\bar{\bar{t}}_1 + l) = -l$. On the other hand, we have $\bar{\bar{s}}_1 = 2k + l - \mu_1$ and $\bar{\bar{s}}_2 = 2k - \mu_1$, thus $\bar{\bar{s}}_2 - \bar{\bar{s}}_1 = -l$, i.e., $\bar{\bar{t}}_2 = \bar{\bar{s}}_2$. Similarly, we can recursively show $\bar{\bar{t}}_i = \bar{\bar{s}}_i$ for all i , $\beta' = k$, $\bar{\bar{t}}_i = \bar{\bar{s}}_i$ for all i . So we have $\mathbf{b}_{1,2} = \mathbf{b}_{2,2}$, and therefore we get the final result $\mathbf{b}_1 = \mathbf{b}_2$.

(iii) The 0-signature of $\mu \otimes x$ is $-2k - \mu_1 \cdot -x_1 + x_2 + C_1$ for some C_1 and that of $1^l \otimes P$ is $-l \cdot -2k - l - \mu_1 + x_1 + x_2 + C_2$ for some C_2 . Here we divide into two cases. Let us first

assume $2k - \mu_1 \geq l$. Then the actions of e_0 on two tensor products look as follows (proceed from left to right):

$$\begin{aligned} \mu \otimes x &: \underbrace{R \cdots R}_{x_1+x_2} \underbrace{L \cdots L}_{2k-\mu_1} \\ 1^l \otimes P &: \underbrace{R \cdots R}_{2k-l-\mu_1+x_1+x_2} \underbrace{L \cdots L}_l \end{aligned}$$

Thus we have $(x_1 + x_2)$ (RR) pairs and l (LL) pairs. Therefore we have $H' = H - (x_1 + x_2) + l$ which gives the desired relation. Next assume $2k - \mu_1 \leq l$. Then we have $(2k - l - \mu_1 + x_1 + x_2)$ (RR) pairs and $(2k - \mu_1)$ (LL) pairs and again we obtain $H' = H - (x_1 + x_2) + l$. \square

6.2. Even r case. Since the proofs are similar to those for the odd r case, we only describe the results.

6.2.1. Calculation in $B^{r,k} \otimes B^{1,l}$. Let $\mu \otimes x \in B^{r,k} \otimes B^{1,l}$ be I_0 -highest and $\mu = \sum_i \mu_i \Lambda_i$. $\mu_i = 0$ unless $1 \leq i \leq r$ and i is even. We also know that the coordinates other than $x_1, x_2, \dots, x_{r+1}, \bar{x}_r, \dots, \bar{x}_4, \bar{x}_2$ are 0 by Proposition 3.1. Let us set $c = (x_1 - \mu_0)_+$ throughout this subsection.

Let us define a word $\mathbf{b}_3 = \mathbf{b}_{3,1} \mathbf{b}_{3,2}$ by

$$\begin{aligned} \mathbf{b}_{3,1} &= 2^{s_2} 3^{s_3} \cdots r^{s_r} (r+1)^\alpha (r+2)^\alpha \cdots n^\alpha (n-1)^\alpha \cdots (r+1)^\alpha r^\alpha \\ &\quad (r-1)^{\bar{s}_{r-1}} (r-2)^{\bar{s}_{r-2}} \cdots 2^{\bar{s}_2}, \\ \mathbf{b}_{3,2} &= 1^{\bar{\bar{s}}_1} 2^{\bar{\bar{s}}_2} 3^{\bar{\bar{s}}_3} \cdots (r-1)^{\bar{\bar{s}}_{r-1}} r^k (r+1)^k \cdots n^k (n-1)^k \cdots (r+1)^k r^k \\ &\quad (r-1)^{\bar{\bar{s}}_{r-1}} (r-2)^{\bar{\bar{s}}_{r-2}} \cdots 2^{\bar{\bar{s}}_2} 1^{\bar{\bar{s}}_1} \end{aligned}$$

where the exponents are defined as follows. For $i = 1, 2, \dots, r/2$,

$$s_{2i} = 2k - \sum_{j=0:\text{even}}^{2i} \mu_j + c + x_{2i+1} + \sum_{j=2:\text{even}}^{2i} \bar{x}_j, \quad s_{2i+1} = s_{2i} - x_{2i+1}.$$

Define $\alpha = k + c + \sum_{j=2:\text{even}}^r \bar{x}_j$. Then $s_r = \alpha + x_{r+1}$. For $i = r/2 - 1, \dots, 2, 1$,

$$\bar{s}_{2i+1} = \bar{s}_{2i} = \alpha - \sum_{j=2i+2:\text{even}}^r \mu_j + \sum_{j=2i+2}^{r+1} x_j.$$

Set $\bar{\bar{s}}_1 = 2k + l - x_1 + c$ and define other $\bar{\bar{s}}_i$ by $\bar{\bar{s}}_{2i} = \bar{\bar{s}}_{2i+1} = 2k - \sum_{j=0:\text{even}}^{2i} \mu_j$ for $i = 1, 2, \dots, r/2 - 1$. Note that $\bar{\bar{s}}_{r-1} = k + \mu_r$. Set $\bar{\bar{s}}_1 = 0$ and define other $\bar{\bar{s}}_i$ by $\bar{\bar{s}}_{2i-1} = \bar{\bar{s}}_{2i-2} = k - \sum_{j=2i:\text{even}}^r \mu_j$ for $i = r/2 - 1, \dots, 3, 2$.

Then we have:

Proposition 6.11. *We have*

$$e_{\mathbf{b}_3} e_0^{2k-\mu_0+c} (\mu \otimes x) = \bar{\mu} \otimes 1^l$$

where $\bar{\mu} = \sum_{i=2:\text{even}}^r \mu_i \Lambda_i + \mu_0 \Lambda_2$.

We remark that $\varepsilon_0(\mu) = 2k - \mu_0$ and $\varphi_0(\mu) = \mu_0$. This nonzero $\varphi_0(\mu)$ is the origin of c in the above formula.

6.2.2. *Calculation in $B^{1,l} \otimes B^{r,k}$.* In this subsection, let P and P' be the \pm -diagrams. As before, corresponding to P and P' , we use the parametrization p_i^* and $p_i'^*$ ($*$ = $\cdot, +, -, \mp$) respectively. Define a word $\mathbf{b}_4 = \mathbf{b}_{4,1}\mathbf{b}_{4,2}$ by

$$\begin{aligned}\mathbf{b}_{4,1} &= 2^{t_2+l} 3^{t_3+l} \dots r^{t_r+l} (r+1)^{\beta+l} (r+2)^{\beta+l} \dots n^{\beta+l} (n-1)^{\beta+l} \dots (r+1)^{\beta+l} r^{\beta+l} \\ &\quad (r-1)^{\bar{t}_{r-1}+l} (r-2)^{\bar{t}_{r-2}+l} \dots 2^{\bar{t}_2+l}, \\ \mathbf{b}_{4,2} &= 1^{\bar{\bar{t}}_1+l} 2^{\bar{\bar{t}}_2} 3^{\bar{\bar{t}}_3} \dots (r-1)^{\bar{\bar{t}}_{r-1}} r^{\beta'} (r+1)^{\beta'} \dots n^{\beta'} (n-1)^{\beta'} \dots (r+1)^{\beta'} r^{\beta'} \\ &\quad (r-1)^{\bar{\bar{t}}_{r-1}} (r-2)^{\bar{\bar{t}}_{r-2}} \dots 2^{\bar{\bar{t}}_2}.\end{aligned}$$

Here the exponents for $\mathbf{b}_{4,1}$ are

$$\begin{aligned}t_2 &= k + \sum_{i=2:\text{even}}^r (p_i^+ - p_i^-) - p_2^+ - p_2^{\mp}, \\ t_{i+1} - t_i &= \begin{cases} -p_{i+1}^+ & (\text{if } i \text{ is odd}) \\ -p_{i+2}^{\mp} & (\text{if } i \text{ is even}), \end{cases} \\ \beta - t_r &= -p_r^{\cdot}, \quad \bar{t}_{r-1} - \beta = -p_r^+ - p_{r-2}^{\cdot}, \\ \bar{t}_i - \bar{t}_{i+1} &= \begin{cases} -p_{i-1}^+ - p_{i+1}^{\cdot} & (\text{if } i \text{ is odd}) \\ 0 & (\text{if } i \text{ is even}). \end{cases}\end{aligned}$$

The exponents for $\mathbf{b}_{4,2}$ are

$$\begin{aligned}\bar{\bar{t}}_1 &= 2k - p_2^{\mp} \\ \bar{\bar{t}}_{i+1} - \bar{\bar{t}}_i &= \begin{cases} -p_{i+3}^{\mp} - p_{i+1}^- - p_{i+1}^+ - p_{i-1}^{\cdot} & (\text{if } i \text{ is odd}) \\ 0 & (\text{if } i \text{ is even}), \end{cases} \\ \beta' - \bar{\bar{t}}_{r-1} &= \bar{\bar{t}}_{r-1} - \beta' = -p_r^{\cdot} - p_r^- - p_r^+ - p_{r-2}^{\cdot}, \\ \bar{\bar{t}}_i - \bar{\bar{t}}_{i+1} &= \begin{cases} -p_{i+3}^{\mp} - p_{i+1}^- - p_{i+1}^+ - p_{i-1}^{\cdot} & (\text{if } i \text{ is odd and } i \neq 1) \\ -p_4^{\mp} - p_2^- - p_2^+ - p_2^{\mp} - p_0^{\cdot} & (\text{if } i = 1) \\ 0 & (\text{if } i \text{ is even}). \end{cases}\end{aligned}$$

Then the result is:

Proposition 6.12. *We have*

$$e_{\mathbf{b}_4} e_0^{\varepsilon_0(P)+l} (1^l \otimes P) = 1^l \otimes P'$$

where

$$\varepsilon_0(P) = p_r^{\cdot} + 2p_r^+ + p_{r-2}^{\cdot} + \sum_{j=2:\text{even}}^{r-2} (p_{j+2}^{\mp} + 2p_j^+ + p_{j-2}^{\cdot}),$$

and P' is related with P as

$$\begin{aligned}p_2'^+ &= p_4^{\mp} + p_2^- + p_2^+ + p_2^{\mp} + p_0^{\cdot}, & p_r'^+ &= p_r^{\cdot} + p_r^- + p_r^+ + p_{r-2}^{\cdot}, \\ p_i'^+ &= p_{i+2}^{\mp} + p_i^- + p_i^+ + p_{i-2}^{\cdot},\end{aligned}$$

where i is an even integer such that $2 < i < r$ and all other $p_i'^* = 0$.

Obviously, P' above is an I_0 -highest weight element.

6.2.3. Reduction to the special case.

Proposition 6.13. *Suppose μ, x and P are related as in Theorem 4.1. Then, with the notions in Propositions 6.11 and 6.12, we have*

- (i) $2k - \mu_0 + c = \varepsilon_0(P) + l$,
- (ii) $\mathbf{b}_3 = \mathbf{b}_4$, and
- (iii) $H(\mu \otimes x) = (x_1 - \mu_0)_+ - l$.

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